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Soft Plus-Product: A New Product for Soft Sets wıth Its Decision-Making

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Abstract

The idea of a soft set offers a comprehensive mathematical basis for managing uncertainty. Since they provide innovative methods for solving issues involving parametric data, soft set operations are important ideas in soft set theory. Here, we provide a novel product operation for soft sets, known as the "soft plus-product," together with all of its algebraic characteristics in relation to various kinds of soft equalities and subsets. Furthermore, by examining the distributions of soft plus-product over various soft set operations, we investigate the relationships between this product and other soft set operations. We end with an example that demonstrates how the method can be used successfully in different scenarios using the int-uni operator and the int-uni decision function for the soft plus-product for the int-uni decision-making method, which chooses a set of optimal elements from the alternatives. This paper is an essential contribution to the literature on soft sets, as the theoretical foundations of soft computing approaches are derived from purely mathematical principles.

Keywords: Soft set, Soft plus-product, Soft subset, Soft equal relations.

1 | Introduction

Most of our traditional tools are precise, trustworthy, and clear for formal modeling, reasoning, and computation. However, uncertainty plays a role in many complex problems in fields such as economics, engineering, social science, medical, and environmental research. Theoretical approaches to uncertainty include the theory of probability, the theory of fuzzy sets [1], the theory of ıntuitionistic fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. Molodstov [2] pointed out in 1999 that these ideas had drawbacks of their own. Molodtsov [2] went on to say that these restrictions could be the consequence of the theory's insufficient parameterization tool.

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Molodstov's [2] soft set theory is very different from conventional theories in this regard. Since soft set theory places no restrictions on the approximate description, it is incredibly useful, easily applicable, and adaptable. Moreover, it is believed that Zadeh's [1] fuzzy set theory, developed in 1965, is an expansion of soft set theory. Other researchers [3–9] created the first novel soft set-based decision-making strategies after Maji et al. [10] conducted the soft set theory to a decision-making problem. Çağman and Enginoğlu [11] developed the fascinating soft set-based decision-making approach known as "uni-int decision-making". Furthermore, Çağman and Enginoğlu [12] presented the soft matrix and created decision-making methods for OR, AND, AND-NOT, and OR-NOT products of the soft matrices. They then used these methods to solve real-world issues, including uncertainty. Since then, soft set theory has been applied extensively and effectively to resolve problems involving decision-making [13– 24].

Several researchers have examined the foundations of soft set theory in recent years. Maji et al. [25] offered a comprehensive theoretical analysis of soft sets, including soft subsets and supersets, equality of soft sets, and soft set operations such as union, intersection, AND, and OR-product. Pei and Miao [26] redefined the terms intersection and soft set subset and looked into the connection between soft sets and information systems. Ali et al. [27] suggested and described novel operations such as the restricted union, restricted intersection, restricted difference, and extended intersection of soft sets. The authors [28-40] then looked at the algebraic structures of the set of all soft sets, suggested better and new approaches, and pointed out certain conceptual mistakes about the foundations of soft set theory that were present in the literature. The study of soft sets has advanced significantly in the last several years. A novel type of difference operation of soft sets was defined by [41]. Stojanovic [42] defined the extended symmetric difference of soft sets and discussed its basic characteristics. Furthermore, new forms of soft set operations have been studied in detail in [43–48].

Soft equal relations and soft subsets are two key concepts in soft set theory. Maji et al. [25] were the first to apply the concept of soft subsets. Pei and Miao [26] and Feng et al. [29] developed the idea of soft subsets, which may be viewed as an extension of Maji's [25] previous definitions. Two new categories of congruence relations and soft equal relations on soft sets were developed by Qin and Hong [49]. Jun and Yang [50] employed extended soft equal relations, which we refer to as J-soft equal relations for consistency's sake, and a broader range of soft subsets to alter Maji's [25] soft distributive laws. The generalized soft distributive principles of soft product operations were further investigated by Jun and Yang in [50]. Liu et al. [52] proposed a brief research note on soft L-subsets and soft L-equal relations inspired by Jun and Yang [50]. One important finding is that not all of the soft equality discussed in the literature is held for distributive norms.
Consequently, by concentrating on soft subsets and the soft products suggested in [25], Feng et al. [51] expanded the research described in [52]. Unlike the notes [52], Feng et al. [51] concentrated on the algebraic characteristics of soft products in terms of the various kinds of soft subsets. They discussed distributional laws, which were thoroughly studied by several scholars, as well as commutative laws, association rules, and other essential characteristics. Along with other related topics, they also offered a theoretical study on how soft products, the AND-product and the OR-product, utilize soft L-subsets. They extensively examined the algebraic features of soft product operations in terms of J-equality and L-equality, and they completed a number of incomplete findings on soft products that had already been published in the literature. On free soft algebras and their related quotient structures, which are commutative semigroups, it was demonstrated that soft L-equal relations are congruent. (See [53–57] for further details on soft equal connections.

To make Molodtsov's [2] soft sets more practical, Çağman and Enginoğlu [11] reformulated the concept and operations of soft sets of Molodtsov [2]. Additionally, they suggested four different kinds of products in soft set theory: AND-product, OR-product, AND-NOT-product, and OR-NOT-product. They suggested a unique decision-making process that selects the best elements from the available possibilities using these new definitions. Lastly, they gave an example showing how the method may be used successfully for a variety of problems, including uncertainty. From a theoretical perspective, Sezgin et al. [58] investigated the AND-product of soft sets, which has long been the basis and a tool utilized by decision-makers in decision-making issues. In [58], the authors thoroughly examined the entire algebraic properties of the AND-product, including idempotent laws, commutative laws, associative laws, and other fundamental properties as regards soft F-subsets and soft M-equality, and compared them to previously obtained properties in terms of soft L-equality and soft J-equality, despite the fact that many scholars have examined the AND-product and its feature in relation to various types of soft equalities, including soft L-equality and soft J-equality. Additionally, they demonstrated that the set of all soft sets over the universe combined with restricted/extended union and AND-product is a commutative hearing with identity in the sense of soft L-equality, and the set of all the soft sets over the universe combined with restricted/extended symmetric difference and AND-product forms a commutative hearing with identity in the sense of soft L-equality. This was accomplished by establishing some new results on the distributive properties of AND-product over restricted, extended, and soft binary soft set operations.

In this study, using Molodtsov's [2] notion of the soft set, we first introduce a novel product for soft sets, which we term the "soft plus-product". We provide its example and thoroughly analyze its algebraic features with respect to several soft subsets and soft equality types, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset /equality. The distributions of soft plus-product over various specific kinds of soft set operations are also obtained. Lastly, we apply the soft decision-making method that selects the best elements from options without the need for fuzzy soft sets or rough sets. We give an example that shows how the method may be

successfully used in a variety of sectors. Since soft sets are an effective mathematical tool for identifying uncertain items and the theoretical underpinnings of soft computing techniques are drawn from purely mathematical concepts, this work attempts to contribute to the literature on soft sets. The structure of this paper is as follows. We review the fundamental ideas of soft set theory in Section 2. The soft plus-product is proposed in Section 3, along with a discussion of its algebraic features in terms of various kinds of soft equalities and soft subsets. In Section 4, we investigate the distributions of soft plus-product over different certain kinds of soft set operations. The int-uni decision operators and function for soft plus-product are applied to solve a decision-making problem in Section 5. There is a little deduction in the conclusion section.

2 | Prelımınarıes

Definition 1 ([1]). Let U be the universal set, E be the parameter, $\mathrm{P(U)}$ be the power set of U and $\mathcal{M} \subseteq \mathrm{E}$. A pair $(\mho.\mathcal{M})$ is called a soft set over U where $\mho$ is a set-valued function such that $\mho: \mathcal{M} \rightarrow \mathrm{P(U)}$.

Although Çağman and Enginoğlu [11] modified Molodstov's [2] concept of soft sets, we continue to use the original definition of soft sets in our work. Throughout this paper, the collections of all the soft sets defined over U is designated $\mathrm{S_E(U)}$. Let $\mathcal{M}$ be a fixed subset of E and $\mathrm{S}_{\mathcal{M}}(\mathrm{U})$ be the collection of all those soft sets over U with the fixed parameter set $\mathcal{M}$. That is, while in the set $\mathrm{S}_{\mathcal{M}}(\mathrm{U})$, there are only soft sets whose parameter sets are $\mathcal{M}$; in the set $\mathrm{S_E(U)}$, there are soft sets whose parameter sets may be any set. From now on, while soft sets will be designated by SS and parameter set by PS, soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

Definition 2 ([27]). Let $(\mho.\mathcal{M})$ be an SS over U. $(\mho.\mathcal{M})$ is called a relative null SS (with respect to the PS $\mathcal{M}$), denoted by $\emptyset_{\mathcal{M}}$, if $\mho(m) = \emptyset$ for all $m \in \mathcal{M}$ and $(\mho.\mathcal{M})$ is called a relative whole SS (with respect to the PS $\mathcal{M}$), denoted by $\mathrm{U}_{\mathcal{M}}$ if $\mho(m) = \mathrm{U}$ for all $m \in \mathcal{M}$. The relative whole SS $\mathrm{U_E}$ with respect to the universe's set of parameters E is called the absolute SS over U.

The empty SS over U is the unique SS over U with an empty PS, represented by $\emptyset_{\emptyset}$. Note $\emptyset_{\emptyset}$ and $\emptyset_{\mathcal{M}}$ are different [31]. In the following, we always consider SSs with non-empty PSs in the universe U, unless otherwise stated.

The concept of soft subset, which we refer to here as soft M-subset to prevent confusion, was initially defined by Maji et al. [25] in the following extremely strict way:

Definition 3 ([25]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. $(\mho.\mathcal{M})$ is called a soft M-subset of $(\mathfrak{F}.\mathcal{D})$ denoted by $(\mho.\mathcal{M}) \tilde{\subseteq}_M (\mathfrak{F}.\mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) = \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ are said to be soft M-equal, denoted by $(\mho.\mathcal{M}) =_M (\mathfrak{F}.\mathcal{D})$, if $(\mho.\mathcal{M}) \tilde{\subseteq}_M (\mathfrak{F}.\mathcal{D})$ and $(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_M (\mho.\mathcal{M})$.

Definition 4 ([26]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. $(\mho.\mathcal{M})$ is called a soft F-subset of $(\mathfrak{F}.\mathcal{D})$ denoted by $(\mho.\mathcal{M}) \tilde{\subseteq}_F (\mathfrak{F}.\mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) \subseteq \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ are said to be soft F-equal, denoted by $(\mho.\mathcal{M}) =_F (\mathfrak{F}.\mathcal{D})$, if $(\mho.\mathcal{M}) \tilde{\subseteq}_F (\mathfrak{F}.\mathcal{D})$ and $(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_F (\mho.\mathcal{M})$.

It should be noted that Pei and Miao [26] initially provided the definitions of soft F-subset and soft F-equal. However, some SS papers regarding soft subsets and soft equalities claimed that Feng et al. [29] provided these definitions first in. As a result, the letter "F" is used to denote this connection.

It was demonstrated in [52] that the soft equal relations =M and =F coincide. In other words, $(\mho.\mathcal{M}) =_M (\mathfrak{F}.\mathcal{D}) \Leftrightarrow (\mho.\mathcal{M}) =_F (\mathfrak{F}.\mathcal{D})$. Since they share the same set of parameters and approximation function, two SSs that meet this soft equivalence are truly identical [52], hence $(\mho.\mathcal{M}) =_M (\mathfrak{F}.\mathcal{D})$ means, in

fact, $(\mho.\mathcal{M}) = (\mathfrak{F}.\mathcal{D})$.

Jun and Yang [50] extended the ideas of F-soft subsets and soft F-equal relations by loosening the restrictions on PSs. We refer to them as soft J-subsets and soft J-equal relations, the initial letter of Jun, even though in

[50] they are named generalized soft subset and generalized soft equal relation.

Definition 5 ([50]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. $(\mho.\mathcal{M})$ is called a soft J-subset of $(\mathfrak{F}.\mathcal{D})$ denoted by $(\mho.\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F}.\mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) \subseteq \mathfrak{F}(d)$. Two SSs $(\mho.\mathcal{M})$

and $(\mathfrak{F}.\mathcal{D})$ are said to be soft J-equal, denoted by $(\mho.\mathcal{M}) =_J (\mathfrak{F}.\mathcal{D})$, if $(\mho.\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F}.\mathcal{D})$ and $(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_J (\mho.\mathcal{M})$.

In [52] and [51], it was shown that $(\mho.\mathcal{M}) \tilde{\subseteq}_M (\mathfrak{F}.\mathcal{D}) \Rightarrow (\mho.\mathcal{M}) \tilde{\subseteq}_F (\mathfrak{F}.\mathcal{D}) \Rightarrow (\mho.\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F}.\mathcal{D})$, but the converse may not be true.

Liu et al. [52] also presented the following new kind of soft subsets (hereafter referred to as soft L-subsets and soft L-equality) that generalize both soft M-subsets and ontology-based soft subsets, inspired by the ideas of soft J-subset [50] and ontology-based soft subsets [30]:

Definition 6 ([52]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. $(\mho.\mathcal{M})$ is called a soft L-subset of $(\mathfrak{F}.\mathcal{D})$ denoted by $(\mho.\mathcal{M}) \tilde{\subseteq}_L (\mathfrak{F}.\mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) = \mathfrak{F}(d)$. Two SSs $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ are said to be soft L-equal, denoted by $(\mho.\mathcal{M}) =_L (\mathfrak{F}.\mathcal{D})$, if $(\mho.\mathcal{M}) \tilde{\subseteq}_L (\mathfrak{F}.\mathcal{D})$ and $(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_J (\mho.\mathcal{M})$.

As regards the relations regarding certain types of soft subsets and soft qualities, $(\mho.\mathcal{M}) \tilde{\subseteq}_M (\mathfrak{F}.\mathcal{D}) \Rightarrow (\mho.\mathcal{M}) \tilde{\subseteq}_L (\mathfrak{F}.\mathcal{D}) \Rightarrow (\mho.\mathcal{M}) \tilde{\subseteq}_J (\mathfrak{F}.\mathcal{D})$ and $(\mho.\mathcal{M}) =_M (\mathfrak{F}.\mathcal{D}) \Rightarrow (\mho.\mathcal{M}) =_L (\mathfrak{F}.\mathcal{D}) \Rightarrow (\mho.\mathcal{M}) =_J (\mathfrak{F}.\mathcal{D})$ [52]. However, the conversation may not be true. Also, it is well-known that $(\mho.\mathcal{M}) =_M (\mathfrak{F}.\mathcal{D})$ if and only if $(\mho.\mathcal{M}) =_F (\mathfrak{F}.\mathcal{D})$.

We may thus deduce that soft M-equality (and so soft F-equality) is the strictest sense, whereas soft J-equality is the weakest soft equal connection. In the middle of these is the idea of the soft L-equal connection [52].

Example 1. Let E=$\{c_1,c_2,c_3,c_4,c_5\}$ be the PS, $\mathcal{M}$ =$\{c_1.c_4\}$ and $\mathcal{D}$=$\{c_1, c_4.c_5\}$ be the subsets of E and U=$\{z_1,z_2,z_3,z_4,z_5\}$ be the initial universe set. Let $(\mho.\mathcal{M})$=$\{(c_1.\{z_1.z_3\}),(c_4.\{z_2.z_3.z_5\})\}$, $(\mathfrak{F}.\mathcal{D})$=$\{(c_1.\{z_1.z_3\}),(c_4.\{z_2,z_3\}),(c_5.\{z_1,z_2.z_3.z_5\})\}$. $(\mathcal{V}.\mathcal{D})$=$\{(c_1.\{z_2.z_3.z_5\}),(c_4.\{z_1,z_3\}),(c_5.\{z_1,z_2.z_3.z_5\})\}$.

Since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (and also $\mho(c_1) \subseteq \mathfrak{F}(c_5)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_5)$, it is obvious that $(\mho.\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}.\mathcal{D})$. However, since $\mho(c_4) \neq \mathfrak{F}(c_1)$, $\mho(c_4) \neq \mathfrak{F}(c_4)$, and $\mho(c_4) \neq \mathfrak{F}(c_5)$, we can deduce that $(\mho.\mathcal{M})$ is not a soft L-subset of $(\mathfrak{F}.\mathcal{D})$. Moreover, as $\mho(c_4) \neq \mathfrak{F}(c_4)$, $(\mho.\mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}.\mathcal{D})$.

Now, since, $\mho(c_1)=\mathcal{V}(c_4)$ and $\mho(c_4)=\mathcal{V}(c_1)$, it is obvious that $(\mho.\mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathcal{V}.\mathcal{D})$. However, as $\mho(c_1) \neq \mathcal{V}(c_1)$, $\mho(c_4) \neq \mathcal{V}(c_4)$, $(\mho.\mathcal{M})$is not again a soft M-subset of $(\mathcal{V}.\mathcal{D})$.

Example 2. Let E=$\{c_1,c_2,c_3,c_4,c_5\}$ be the PS, $\mathcal{M}=\{c_1.c_4\}$ and $\mathcal{D}=\{c_1,c_4.c_5\}$ be the subsets of E and U=$\{z_1,z_2,z_3,z_4,z_5\}$ be the initial universe set. Let $(\mho.\mathcal{M})=\{(c_1.\{z_1.z_3\}),(c_4.\{z_1.z_2.z_3.z_5\})\}$, $(\mathfrak{F}.\mathcal{D})=\{(c_1.\{z_1.z_2.z_3\}),(c_4.\{z_1.z_2.z_3.z_5\}),(c_5.\{z_1\})\}$.

Since $\mho(c_1) \neq \mathfrak{F}(c_1)$, $\mho(c_1) \neq \mathfrak{F}(c_4)$ and $\mho(c_1) \neq \mathfrak{F}(c_5)$, it is obvious that $(\mho.\mathcal{M}) \neq_{\mathrm{L}} (\mathfrak{F}.\mathcal{D})$. However, since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (moreover $\mho(c_1) \subseteq \mathfrak{F}(c_4)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_4)$, we can deduce that $(\mho.\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}.\mathcal{D})$. Moreover, since $\mathfrak{F}(c_1) \subseteq \mho(c_4)$ and $\mathfrak{F}(c_4) \subseteq \mho(c_4)$, and $\mathfrak{F}(c_5) \subseteq \mho(c_1)$. we can deduce that $(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho.\mathcal{M})$. Therefore, $(\mho.\mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}.\mathcal{D})$. As $\mho(c_1) \neq \mathfrak{F}(c_1)$ and $\mho(c_4) \neq \mathfrak{F}(c_4)$, it is obvious that $(\mho.\mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}.\mathcal{D})$.

For more on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and some other existing definitions of soft subsets and soft equal relations in the literature, we refer to [49–57].

Definition 7 ([27]). Let $(\mho.\mathcal{M})$ be an SS over U. The relative complement of an SS Let $(\mho.\mathcal{M})$, denoted by $(\mho.\mathcal{M})^{\mathrm{r}}$. is defined by $(\mho.\mathcal{M})^{\mathrm{r}} = (\mho^{\mathrm{r}}.\mathcal{M})$, where $\mho^{\mathrm{r}}: \mathcal{M} \to \mathrm{P(U)}$ is a mapping given by $(\mho.\mathcal{M})^{\mathrm{r}} = \mathrm{U} \setminus \mho(m)$ for all $m \in \mathcal{M}$. From now on, $\mathrm{U}\setminus\mho(m)=[\mho(m)]'$ is designated by $\mho'(m)$ for the sake of designation.

Definition 8 ([25]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. The AND-product ($\wedge$-product) of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ is denoted by $(\mho.\mathcal{M})\wedge(\mathfrak{F}.\mathcal{D})$, and is defined by $(\mho.\mathcal{M})\wedge(\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mho(m.d) = \mho(m) \cap \mathfrak{F}(d)$.

Definition 9 ([25]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. The OR-product ($\vee$-product) of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ is denoted by $(\mho.\mathcal{M}) \vee (\mathfrak{F}.\mathcal{D})$, and is defined by $(\mho.\mathcal{M}) \vee (\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mho(m.d) = \mho(m) \cup \mathfrak{F}(d)$.

Çağman [59] defined inclusive complement and exclusive complements as a novel idea in set theory and investigated the connections between these two by contrasting them. In [59], these new concepts were also applied to group theory and Sezgin et al. [60] were inspired by the study.

Definition 10 ([59]). Let A and B be two subsets of the universe. Then, the B-inclusive complement of A is defined by $\mathrm{A} + \mathrm{B} := \mathrm{A}' \cup \mathrm{B}$.

Then, plus operation was applied to soft set theory to propose new soft set operations [61–64]. Let's use "$\circledast$" to stand for set operations like $\cap$. $\cup$. $\setminus.\Delta$. The following definitions are given for restricted, extended and soft binary piecewise operations.

Definition 11 ([27]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. The restricted $\circledast$ operation of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$, denoted by $(\mho.\mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}.\mathcal{D})$ is defined by $(\mho.\mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M} \cap \mathcal{D})$, where $\mathcal{J} = \mathcal{M} \cap \mathcal{D}$ and if $\mathcal{J} \neq \emptyset$, then for all $\mathrm{j} \in \mathcal{J}$, $\mho(\mathrm{j}) = \mho(\mathrm{j}) \circledast \mathfrak{F}(\mathrm{j})$; if $\mathcal{J} = \emptyset$, then $(\mho.\mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}.\mathcal{D}) = \emptyset_{\emptyset}$.

Definition 12 ([27], [42]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. The extended $\circledast$ operation of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$, denoted by $(\mho.\mathcal{M}) \circledast_{\varepsilon} (\mathfrak{F}.\mathcal{D})$ is defined by $(\mho.\mathcal{M}) \circledast_{\varepsilon} (\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M} \cup \mathcal{D})$, where $\mathcal{J} = \mathcal{M} \cup \mathcal{D}$ and then for all $\mathrm{j} \in \mathcal{J}$,

$$\mho(\mathrm{j}) = \begin{cases} \mho(\mathrm{j}), & \mathrm{j} \in \mathcal{M}\setminus\mathcal{D}, \\ \mathfrak{F}(\mathrm{j}), & \mathrm{j} \in \mathcal{D}\setminus\mathcal{M}, \\ \mho(\mathrm{j}) \circledast \mathfrak{F}(\mathrm{j}), & \mathrm{j} \in \mathcal{M} \cap \mathcal{D}. \end{cases}$$

Definition 13 ([43]). Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be two SSs over U. The soft binary piecewise $\circledast$ operation of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$, denoted by $(\mho.\mathcal{M}) \widetilde{\circledast} (\mathfrak{F}.\mathcal{D})$, is defined by $(\mho.\mathcal{M}) \widetilde{\circledast} (\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M})$, where for all $j \in \mathcal{M}$,

$$\mho(j) = \begin{cases} \mho(j), & j \in \mathcal{M}\backslash\mathcal{D}, \\ \mho(j) \circledast \mathfrak{F}(j), & j \in \mathcal{M} \cap \mathcal{D}. \end{cases}$$

For more about soft algebraic structures of soft sets, we refer to [65-90].

3 | Soft Plus-Product and Its Algebraic Properties

In this subsection, we introduce a new product for soft sets called soft plus products. We give its example and examine its algebraic properties in detail depth in terms of specific kinds of soft equalities and soft subsets.

Definition 14. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. The soft plus product of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$, denoted by $(\mho.\mathcal{M})\mathrm{V}_+(\mathfrak{F}.\mathcal{D})$, is defined by $(\mho.\mathcal{M})\mathrm{V}_+(\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$,

$$\mho(m.d) = \mho(m) + \mathfrak{F}(d).$$

Here,

$$\mho(m) + \mathfrak{F}(d) = \mho'(m) \cup \mathfrak{F}(d).$$

Example 3. Assume that $\mathrm{E} = \{e_1.e_2.e_3.e_4.e_5\}$ be the PS, $\mathcal{M} = \{e_1.e_2.e_3\}$ and $\mathcal{D} = \{e_1.e_4.e_5\}$ be the subsets of E, $\mathrm{U} = \{h_1.h_2.h_3.h_4.h_5.h_6\}$ be the universal set, the SSs $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be over U such that

$$(\mho.\mathcal{M}) = \{(e_1.\{h_1.h_2.h_3.h_5\}).(e_2.\{h_1.h_2.h_3\}).(e_3.\{h_4.h_5.h_6\})\},$$
$$(\mathfrak{F}.\mathcal{D}) = \{(e_1.\{h_6\}).(e_4.\{h_2.h_3.h_5\}).(e_5.\{h_2\})\}.$$

Let $(\mho,\mathcal{M})\mathrm{V}_+(\mathfrak{F},\mathcal{D}) = (\mho,\mathcal{M}\mathrm{x}\mathcal{D})$. Then,

$(\mho.\mathcal{M}\mathrm{x}\mathcal{D})$
$= \{((e_1.e_1).\{h_4.h_6\}).((e_1.e_4).\{h_2.h_3.h_4.h_5.h_6\}).((e_1.e_5).\{h_2.h_4.h_6\}).((e_2.e_1).\{h_4.h_5.h_6\}).$

$((e_2.e_4).\{h_2.h_3.h_4.h_5.h_6\}).((e_2.e_5).\{h_2.h_4.h_5.h_6\}).((e_3.e_1).\{h_1.h_2.h_3.h_6\}).((e_3.e_4).\{h_1.h_2.h_3.h_5\}).$

$\big((e_3.e_5).\{h_1.h_2.h_3\}\big)\}.$

Here, the table method can be used as it is more convenient than writing in the list method format:

Table 1. The comparison and analysis of the data is presented in this table.

$(\mho,\mathcal{M})\mathrm{V}_+(\mathfrak{F},\mathcal{D})$	e_1	e_4	e_5
e_1	$\{h_4,h_6\}$	$\{h_2,h_3,h_4,h_5,h_6\}$	$\{h_2,h_4,h_6\}$
e_2	$\{h_4,h_5,h_6\}$	$\{h_2,h_3,h_4,h_5,h_6\}$	$\{h_2,h_4,h_5,h_6\}$
e_3	$\{h_1,h_2,h_3,h_6\}$	$\{h_1,h_2,h_3,h_5\}$	$\{h_1,h_2,h_3\}$

Proposition 1. V_+-product is closed in $S_E(U)$.

Proof: it is obvious that V_+-product is a binary operation in $S_E(U)$. In fact, let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then,

$$\mathrm{V}_+: S_E(U) \times S_E(U) \longrightarrow S_E(U),$$
$$\big((\mho.\mathcal{M}).(\mathfrak{F}.\mathcal{D})\big) \longrightarrow (\mho.\mathcal{M})\mathrm{V}_+(\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M}\mathrm{x}\mathcal{D}) = (\mho.\mathcal{J}).$$

That is, $(\text{Ꮙ}.\mathcal{J})$ is an SS over U. since the set $S_E(U)$ contains all the SS over U. Here, note that the set $S_{\mathcal{M}}(U)$ is not closed under V_+-product, since if $(\mho.\mathcal{M}).(\mathfrak{F}.\mathcal{M})$ are the elements of $S_{\mathcal{M}}(U)$, $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{M})$ is an element of $S_{\mathcal{M}x\mathcal{M}}(U)$ not $S_{\mathcal{M}}(U)$.

Proposition 2. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ and $(\text{Ꮙ}.\mathcal{J})$ be SSs over U. Then,

$$(\mho.\mathcal{M})V_+[(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J})] \neq_M [(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D})]V_+(\text{Ꮙ}.\mathcal{J}).$$

Thus, V_+-product is not associative in $S_E(U)$.

Proof: in order to show that V_+-product is not associative in $S_E(U)$, we provided an example: Let $E = \{e_1.e_2.e_3.e_4\}$ be the PS, $\mathcal{M} = \{e_2.e_3\}$, $\mathcal{D} = \{e_1\}$ and $\mathcal{J} = \{e_4\}$ be the subsets of E, $U = \{h_1.h_2.h_3.h_4.h_5\}$ be the universal set, and $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ ve $(\text{Ꮙ}.\mathcal{J})$ be SSs over U such that $(\mho.\mathcal{M}) = \{(e_2.\{h_3.h_4\}).(e_3.\{h_1\})\}$, $(\mathfrak{F}.\mathcal{D}) = \{(e_1.\emptyset)\}$ and $(\text{Ꮙ}.\mathcal{J}) = \{(e_4.\{h_1.h_3.h_5\})\}$ We show that $(\mho.\mathcal{M})V_+[(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J})] \neq_M [(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D})]V_+(\text{Ꮙ}.\mathcal{J})$. Let $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) = (\text{Ꮳ}.\mathcal{D}x\mathcal{J})$, Then,

$$(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) = (\text{Ꮳ}.\mathcal{D}x\mathcal{J}) = \{((e_1.e_4).U)\},$$

and let $(\mho.\mathcal{M})V_+(\text{Ꮳ}.\mathcal{D}x\mathcal{J}) = (\mathfrak{X}.\mathcal{M}x(\mathcal{D}x\mathcal{J}))$. Thus,

$$(\mathfrak{X}.\mathcal{M}x(\mathcal{D}x\mathcal{J})) = \{((e_2.(e_1.e_4)).u).((e_3.(e_1.e_4)).U)\}.$$

Assume that $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) = (\text{ƻ}.\mathcal{M}x\mathcal{D})$. Thereby,

$$(\text{ƻ}.\mathcal{M}x\mathcal{D}) = \{((e_2.e_1).\{h_1.h_2.h_5\}).((e_3.e_1).\{h_2.h_3.h_4.h_5\})\}.$$

Suppose that $(\text{ƻ}.\mathcal{M}x\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) = (\text{ღ}.(\mathcal{M}x\mathcal{D})x\mathcal{J})$. Therefore,

$$(\text{ღ}.(\mathcal{M}x\mathcal{D})x\mathcal{J}) = \{(((e_2.e_1).e_4).\{h_4\}).(((e_3.e_1).e_4).\emptyset)\}.$$

It is seen that $(\mathfrak{X}.\mathcal{M}x(\mathcal{D}x\mathcal{J})) \neq_M (\text{ღ}.(\mathcal{M}x\mathcal{D})x\mathcal{J})$. It is obvious that $(\mathfrak{X}.\mathcal{M}x(\mathcal{D}x\mathcal{J})) \neq_L (\text{ღ}.(\mathcal{M}x\mathcal{D})x\mathcal{J})$ and $(\mathfrak{X}.\mathcal{M}x(\mathcal{D}x\mathcal{J})) \neq_J (\text{ღ}.(\mathcal{M}x\mathcal{D})x\mathcal{J})$.

Proposition 3. Let $(\mathfrak{F}.\mathcal{D})$ and $(\text{Ꮙ}.\mathcal{J})$ be SSs over U. Then, $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) \neq_M (\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D})$. That is, V_+-product is not commutative in $S_E(U)$.

Proof: let $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) = (\text{Ꮳ}.\mathcal{D}x\mathcal{J})$ and $(\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D})=(\mathfrak{X}.\mathcal{J}x\mathcal{D})$. Since $\mathcal{D}x\mathcal{J} \neq \mathcal{J}x\mathcal{D}$, the rest of the proof is obvious.

Proposition 4. Let $(\mathfrak{F}.\mathcal{D})$ and $(\text{Ꮙ}.\mathcal{J})$ be SSs over U. Then, $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) \neq_J (\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D})$. That is, Λ_+-product is not commutative in $S_E(U)$ under J-equlity.

Proof: in order to show that Λ_+-product is not commutative in $S_E(U)$ under J-equality, we provide an example. Let $E = \{e_1.e_2.e_3.e_4.e_5\}$ be the PS, $\mathcal{D} = \{e_3.e_5\}$ and $\mathcal{J} = \{e_1\}$ be the subsets of E, $U = \{h_1.h_2.h_3.h_4.h_5\}$ be the universal set, $(\mathfrak{F}.\mathcal{D})$ and $(\text{Ꮙ}.\mathcal{J})$ be SSs over U such that $(\mathfrak{F}.\mathcal{D}) = \{(e_3.\{h_1.h_2\}).(e_5.U)\}$ and $(\text{Ꮙ}.\mathcal{J}) = \{(e_1.\{h_5\})\}$. We show that $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) \neq_J (\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D})$. Let $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) = (\text{Ꮳ}.\mathcal{D}x\mathcal{J})$. Then,

$$(\text{Ꮳ}.\mathcal{D}x\mathcal{J}) = \{((e_3.e_1).\{h_3.h_4.h_5\}).((e_5.e_1).\{h_5\})\}.$$

Suppose that $(\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D}) = (\text{ƻ}.\mathcal{J}x\mathcal{D})$. Then,

$$(\text{ƻ}.\mathcal{J}x\mathcal{D}) = \{((e_1.e_3).\{h_1.h_2.h_3.h_4\}).((e_1.e_5).U)\}.$$

Thus, $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) \neq_J (\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D})$. Moreover, $(\mathfrak{F}.\mathcal{D})V_+(\text{Ꮙ}.\mathcal{J}) \neq_L (\text{Ꮙ}.\mathcal{J})V_+(\mathfrak{F}.\mathcal{D})$.

Proposition 5. Let $(\mathfrak{F}.\mathcal{D})$ be an SS over U. Then, $(\mathfrak{F}.\mathcal{D})V_+\emptyset_\emptyset =_M \emptyset_\emptyset V_+(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$. That is, $\emptyset_\emptyset$-the empty SS-is the absorbing element of V_+-product in $S_E(U)$.

Proof: let $\emptyset_\emptyset = (\varsigma.\emptyset)$ and $(\mathfrak{F}.\mathcal{D})\vee_+\emptyset_\emptyset = (\mathfrak{F}.\mathcal{D})\vee_+(\varsigma.\emptyset) = (\wp.\mathcal{D}\mathrm{x}\emptyset) = (\wp.\emptyset)$. Since the only SS whose PS is $\emptyset_\emptyset$, $(\wp.\emptyset) = \emptyset_\emptyset$. One can similarly show that $\emptyset_\emptyset \vee_+ (\mathfrak{F}.\mathcal{D}) =_\mathrm{M} \emptyset_\emptyset$.

Proposition 6. Let $(\mho.\mathcal{M})$ be an SS over U. Then, $(\mho.\mathcal{M})\vee_+\mathrm{U}_\mathcal{M} =_\mathrm{L} \mathrm{U}_\mathcal{M}$. That is, $\mathrm{U}_\mathcal{M}$ is the right absorbing element of $\vee_+$-product in $\mathrm{S}_\mathcal{M}(\mathrm{U})$ under L-equality.

Proof: let $\mathrm{U}_\mathcal{M} = (\mathcal{V}.\mathcal{M})$ and $(\mho.\mathcal{M})\vee_+(\mathcal{V}.\mathcal{M}) = (\mathfrak{z}.\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \mathrm{U}$ and for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{z}(m.d) = \mho'(m) \cup \mathcal{V}(d)=\mho'(m) \cup \mathrm{U} = \mathrm{U}$. Since, for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{z}(m.d) = \mathrm{U} = \mathcal{V}(m)$, implying that $(\mho.\mathcal{M})\vee_+\mathrm{U}_\mathcal{M} \widetilde{\subseteq}_\mathrm{L} \mathrm{U}_\mathcal{M}$. Moreover, for all $m \in \mathcal{M}$, there exists $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$ such that $\mathcal{V}(m) = \mathrm{U} = \mathfrak{z}(m.d)$, implying that $\mathrm{U}_\mathcal{M} \widetilde{\subseteq}_\mathrm{L} (\mho.\mathcal{M})\vee_+\mathrm{U}_\mathcal{M}$. Thereby, $(\mho.\mathcal{M})\vee_+\mathrm{U}_\mathcal{M} =_\mathrm{L} \mathrm{U}_\mathcal{M}$.

Proposition 7. Let $(\mho.\mathcal{M})$ be an SS over U. Then, $\mathrm{U}_\mathcal{M}\vee_+(\mho.\mathcal{M}) =_\mathrm{L} (\mho.\mathcal{M})$. That is, $\mathrm{U}_\mathcal{M}$ is the left identity element of $\vee_+$-product in $\mathrm{S}_\mathcal{M}(\mathrm{U})$ under L-equality.

Proof: let $\mathrm{U}_\mathcal{M} = (\mathcal{V}.\mathcal{M})$ and $(\mathcal{V}.\mathcal{M})\vee_+(\mho.\mathcal{M}) = (\mathfrak{z}.\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \mathrm{U}$ and for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{z}(m.d) = \mathcal{V}'(m) \cup \mho(d)=\emptyset \cup \mho(d) = \mho(d)$. Since, for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $d \in \mathcal{M}$ such that $\mathfrak{z}(m.d) = \mho(d)$, implying that $\mathrm{U}_\mathcal{M}\vee_+(\mho.\mathcal{M}) \widetilde{\subseteq}_\mathrm{L} (\mho.\mathcal{M})$. Moreover, for all $d \in \mathcal{M}$, there exists $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$ such that $\mho(d) = \mathfrak{z}(m.d)$, implying that $(\mho.\mathcal{M}) \widetilde{\subseteq}_\mathrm{L} \mathrm{U}_\mathcal{M}\vee_+(\mho.\mathcal{M})$. Thereby, $\mathrm{U}_\mathcal{M}\vee_+(\mho.\mathcal{M}) =_\mathrm{L} (\mho.\mathcal{M})$.

Proposition 8. Let $(\mathfrak{F}.\mathcal{D})$ be an SS over U. Then, $(\mathfrak{F}.\mathcal{D})\vee_+\emptyset_\mathcal{D} =_\mathrm{M} (\mathfrak{F}.\mathcal{D}\mathrm{x}\mathcal{D})^\mathrm{r}$ ve $\emptyset_\mathcal{D} \vee_+(\mathfrak{F}.\mathcal{D}) =_\mathrm{M} \mathrm{U}_{\mathcal{D}\mathrm{x}\mathcal{D}}$.

Proof: let $\emptyset_\mathcal{D} = (\varsigma.\mathcal{D})$. Then, for all $d \in \mathcal{D}. \varsigma(d) = \emptyset$. Let $(\mathfrak{F}.\mathcal{D})\vee_+\emptyset_\mathcal{D} = (\mathfrak{F}.\mathcal{D})\vee_+(\varsigma.\mathcal{D}) = (\wp.\mathcal{D}\mathrm{x}\mathcal{D})$. Thus, for all $(d.m) \in \mathcal{D}\mathrm{x}\mathcal{D}. \wp(d.m) = \mathfrak{F}'(d) \cup \varsigma(m) = \mathfrak{F}'(d) \cup \emptyset = \mathfrak{F}'(d)$, implying that $(\wp.\mathcal{D}\mathrm{x}\mathcal{D}) = (\mathfrak{F}.\mathcal{D}\mathrm{x}\mathcal{D})^\mathrm{r}$. Similarly, let $\emptyset_\mathcal{D} \vee_+(\mathfrak{F}.\mathcal{D}) =_\mathrm{M} (\mathfrak{X}.\mathcal{D}\mathrm{x}\mathcal{D})$. Then, since for all $(d.m) \in \mathcal{D}\mathrm{x}\mathcal{D}$. $\mathfrak{X}(d.m) = \varsigma'(d) \cup \mathfrak{F}(m) = \emptyset' \cup \mathfrak{F}(m) = \mathrm{U} \cup \mathfrak{F}(m) = \mathrm{U}$, implying that $(\mathfrak{X}.\mathcal{D}\mathrm{x}\mathcal{D}) = \mathrm{U}_{\mathcal{D}\mathrm{x}\mathcal{D}}$,

Proposition 9. Let $(\mathfrak{F}.\mathcal{D})$ an SS over U. Then, $(\mathfrak{F}.\mathcal{D})\vee_+\mathrm{U}_\mathcal{D} =_\mathrm{M} \mathrm{U}_{\mathcal{D}\mathrm{x}\mathcal{D}}$ ve $\mathrm{U}_\mathcal{D}\vee_+(\mathfrak{F}.\mathcal{D}) =_\mathrm{M} (\mathfrak{F}.\mathcal{D}\mathrm{x}\mathcal{D})$.

Proof: let $\mathrm{U}_\mathcal{D} = (\mathcal{V}.\mathcal{D})$. Then, for all $d \in \mathcal{D}, \mathcal{V}(d) = \mathrm{U}$. Let $(\mathfrak{F}.\mathcal{D})\vee_+\mathrm{U}_\mathcal{D} = (\mathfrak{F}.\mathcal{D})\vee_+(\mathcal{V}.\mathcal{D}) = (\mathfrak{X}.\mathcal{D}\mathrm{x}\mathcal{D})$. Thus, for all $(d.m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $\mathfrak{X}(d.m) = \mathfrak{F}'(d) \cup \mathcal{V}(m) = \mathfrak{F}'(d) \cup \mathrm{U} = \mathrm{U}$, implying that $(\mathfrak{X}.\mathcal{D}\mathrm{x}\mathcal{D}) = \mathrm{U}_{\mathcal{D}\mathrm{x}\mathcal{D}}$. Let $\mathrm{U}_\mathcal{D}\vee_+(\mathfrak{F}.\mathcal{D}) =_\mathrm{M} (\wp.\mathcal{D}\mathrm{x}\mathcal{D})$. Then, for all $(d.m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $\wp(d.m) = \mathcal{V}'(d) \cup \mathfrak{F}(m) = \mathrm{U}' \cup \mathfrak{F}(m) = \emptyset \cup \mathfrak{F}(m) = \mathfrak{F}(m)$. Thus, $(\wp.\mathcal{D}\mathrm{x}\mathcal{D}) = (\mathfrak{F}.\mathcal{D}\mathrm{x}\mathcal{D})$.

Proposition 10. Let $(\mho.\mathcal{M})$ be an SS over U. Then, $(\mho.\mathcal{M}) \widetilde{\subseteq}_\mathrm{J} (\mho.\mathcal{M})\vee_+(\mho.\mathcal{M})$, That is, $\vee_+$-product is not idempotent in $\mathrm{S}_\mathrm{E}(\mathrm{U})$ under J-equality.

Proof: let $(\mho.\mathcal{M})\vee_+(\mho.\mathcal{M}) = (\mathfrak{F}.\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{F}(m.d) = \mho'(m) \cup \mho(d)$. Since for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{M}$. there exists $d \in \mathcal{M}$ such that $\mathfrak{F}(m.d) = \mho(d) \subseteq \mho'(m) \cup \mho(d)$. $(\mho.\mathcal{M}) \widetilde{\subseteq}_\mathrm{J} (\mathfrak{F}.\mathcal{M}\mathrm{x}\mathcal{M})$ is obtained.

Proposition 11. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U, Then, $(\mathfrak{F}.\mathcal{D}) \widetilde{\subseteq}_\mathrm{J} (\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})$ and $(\mho.\mathcal{M})^\mathrm{r} \widetilde{\subseteq}_\mathrm{J} (\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})$.

Proof: let $(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D}) = (\mathcal{V}.\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$. $\mathcal{V}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Since for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, there exists $d \in \mathcal{D}$ such that $\mathfrak{F}(d) \subseteq \mho'(m) \cup \mathfrak{F}(d)$, $(\mathfrak{F}.\mathcal{D}) \widetilde{\subseteq}_\mathrm{J} (\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})$. Similarly, since for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$. there exists $m \in \mathcal{M}$ such that $\mho'(m) \subseteq \mho'(m) \cup \mathfrak{F}(d)$, $(\mho.\mathcal{M})^\mathrm{r} \widetilde{\subseteq}_\mathrm{J} (\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})$.

Proposition 12. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $[(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})]^\mathrm{r} = (\mho.\mathcal{M})^\mathrm{r}\wedge_\gamma(\mathfrak{F}.\mathcal{D})^\mathrm{r}$.

Proof: let $(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D}) = (\mathcal{V}.\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Thus, $\mathcal{V}'(m.d) = \mho(m) \cap \mathfrak{F}'(d) = (\mho')'(m) \cap \mathfrak{F}'(d)$. Hence, $(\mathcal{V}'.\mathcal{M}\mathrm{X}\mathcal{D}) = (\mho.\mathcal{M})^\mathrm{r}\wedge_\gamma(\mathfrak{F}.\mathcal{D})^\mathrm{r}$. (For $\wedge_\gamma$-product, please see [91].)

Proposition 13. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $(\mho.\mathcal{M})\wedge_\gamma(\mathfrak{F}.\mathcal{D}) \widetilde{\subseteq}_\mathrm{F} (\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})$.

Proof: let $(\mho.\mathcal{M})\wedge_{\gamma}(\mathfrak{F}.\mathcal{D}) = (\mathfrak{V}.\mathcal{M}x\mathcal{D})$ and $(\mho.\mathcal{M})+_{*}(\mathfrak{F}.\mathcal{D}) = (\mho.\mathcal{M}x\mathcal{D})$. Then, for all $(m.d) \in \mathcal{M}x\mathcal{D}$. $\mathfrak{V}(m.d) = \mho'(m) \cap \mathfrak{F}(d)$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mho(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Thus, for all $(m.d) \in \mathcal{M}x\mathcal{D}$. $\mathfrak{V}(m.d) = \mho'(m) \cap \mathfrak{F}(d) \subseteq \mho'(m) \cup \mathfrak{F}(d) = \mho(m.d)$. This completes the proof.

Proposition 14. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ and $(\mathfrak{V}.\mathcal{J})$ be SSs over U. If $(\mho.\mathcal{M}) \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D})$. then $(\mathfrak{V}.\mathcal{J})V_{+}(\mho.\mathcal{M}) \tilde{\subseteq}_{F} (\mathfrak{V}.\mathcal{J})V_{+}(\mathfrak{F}.\mathcal{D})$.

Proof: let $\mho.\mathcal{M}) \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D}$. Then, $\mathcal{M} \subseteq \mathcal{D}$ and for all $m \in \mathcal{M}$, $\mho(m) \subseteq \mathfrak{F}(m)$. Thus, $\mathcal{J}xx\mathcal{M} \subseteq \mathcal{J}x\mathcal{D}$ and for all $(j.m) \in \mathcal{J}x\mathcal{M}$, $\mathfrak{V}'(j) \cup \mho(m) \subseteq \mathfrak{V}'(j) \cup \mathfrak{F}(m)$. This completes the proof.

Proposition 15. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$, $(\mathfrak{V}.\mathcal{J})$ and $(\mho.\mathfrak{X})$ be SSs over U. If $(\mho.\mathcal{M})^{r} \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D})^{r}$ and $(\mathfrak{V}.\mathcal{J}) \tilde{\subseteq}_{F} (\mho.\mathfrak{X})$, then $(\mho.\mathcal{M})V_{+}(\mathfrak{V}.\mathcal{J}) \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D})V_{+}(\mho.\mathfrak{X})$.

Proof: let $(\mho.\mathcal{M})^{r} \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D})^{r}$ ve $(\mathfrak{V}.\mathcal{J}) \tilde{\subseteq}_{F} (\mho.\mathfrak{X})$. Then, $\mathcal{M} \subseteq \mathcal{D}$, $\mathcal{J} \subseteq \mathfrak{X}$, for all $m \in \mathcal{M}$. $\mho'(m) \subseteq \mathfrak{F}'(m)$ and for all $j \in \mathcal{J}$. $\mathfrak{V}(j) \subseteq \mho(j)$. Thus, $\mathcal{M}x\mathcal{J} \subseteq \mathcal{D}x\mathfrak{X}$, for all $(m.j) \in \mathcal{M}x\mathcal{J}$, $\mho'(m) \cup \mathfrak{V}(j) \subseteq \mathfrak{F}'(m) \cup \mho(j)$ completing the proof.

Proposition 16. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{M})$, $(\mho.\mathcal{M})$ and $(\mathfrak{V}.\mathcal{M})$ be SSs over U. If $(\mho.\mathcal{M}) \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{M})$ and $(\mathfrak{V}.\mathcal{M}) \tilde{\subseteq}_{F} (\mho.\mathcal{M})$, then $(\mho.\mathcal{M})V_{+}(\mho.\mathcal{M}) \tilde{\subseteq}_{F} (\mathfrak{V}.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{M})$.

Proof: let $(\mho.\mathcal{M}) \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{M})$ and $(\mathfrak{V}.\mathcal{M}) \tilde{\subseteq}_{F} (\mho.\mathcal{M})$. Thus, for all $m \in \mathcal{M}.\mho(m) \subseteq \mathfrak{F}(m)$ and for all $j \in \mathcal{M}$. $\mathfrak{V}(j) \subseteq \mho(j)$. Hence, for all $(m.j) \in \mathcal{M}x\mathcal{M}$, $\mho'(j) \cup \mho(m) \subseteq \mathfrak{V}'(j) \cup \mathfrak{F}(m)$. This completes the proof.

Proposition 17. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $\emptyset_{\mathcal{M}x\mathcal{D}} \tilde{\subseteq}_{F} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$ and $\emptyset_{\mathcal{D}x\mathcal{M}} \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D})V_{+}(\mho.\mathcal{M})$.

Proof: let $\emptyset_{\mathcal{M}x\mathcal{D}} = (\mathfrak{V}.\mathcal{M}x\mathcal{D})$ and $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) = (\mathfrak{z}.\mathcal{M}x\mathcal{D})$. Then, for $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{V}(m.d) = \emptyset$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{z}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Since $\mathcal{M}x\mathcal{D} \subseteq \mathcal{M}x\mathcal{D}$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{V}(m.d) = \emptyset \subseteq \mho'(m) \cup \mathfrak{F}(d).\mathfrak{z}(m.d)$. $\emptyset_{\mathcal{M}x\mathcal{D}} \tilde{\subseteq}_{F} (\mho.\mathcal{M})\wedge_{\theta}(\mathfrak{F}.\mathcal{D})$ is obtained. Similarly, $\emptyset_{\mathcal{D}x\mathcal{M}} \tilde{\subseteq}_{F} (\mathfrak{F}.\mathcal{D})V_{+}(\mho.\mathcal{M})$ can be shown.

Proposition 18. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{J} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$, $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{J} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$ and $\emptyset_{E} \tilde{\subseteq}_{J} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$.

Proof: Let $\emptyset_{\mathcal{M}} = (\mathfrak{V}.\mathcal{M})$ and $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) = (\mathfrak{z}.\mathcal{M}x\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathfrak{V}(m) = \emptyset$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{z}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Since for all $m \in \mathcal{M}$, there exists $(m.d) \in \mathcal{M}x\mathcal{D}$ such that $\mathfrak{V}(m) = \emptyset \subseteq \mho'(m) \cup \mathfrak{F}(d) = \mathfrak{z}(m.d)$. $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{J} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$ is obtained. One can similarly show that $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{J} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$ and $\emptyset_{E} \tilde{\subseteq}_{J} (\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D})$.

Proposition 19. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ ve SSs over U. Then, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{F} U_{\mathcal{M}x\mathcal{D}}$ and $(\mathfrak{F}.\mathcal{D})\wedge_{\theta}(\mho.\mathcal{M}) \tilde{\subseteq}_{F} U_{\mathcal{D}x\mathcal{M}}$.

Proof: let $U_{\mathcal{M}x\mathcal{D}} = (\mathfrak{V}.\mathcal{M}x\mathcal{D})$ and $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) = (\mathfrak{z}.\mathcal{M}x\mathcal{D})$. Then, for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{V}(m.d) = U$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{z}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Since $\mathcal{M}x\mathcal{D} \subseteq \mathcal{M}x\mathcal{D}$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{z}(m.d) = \mho'(m) \cup \mathfrak{F}(d) \subseteq U = \mathfrak{V}(m.d)$, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{F} U_{\mathcal{M}x\mathcal{D}}$ is obtained. One can similarly show that $(\mathfrak{F}.\mathcal{D})V_{+}(\mho.\mathcal{M}) \tilde{\subseteq}_{F} U_{\mathcal{D}x\mathcal{M}}$.

Proposition 20. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{M}}$, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{D}}$.

Proof: let $U_{\mathcal{M}} = (\mathfrak{V}.\mathcal{M})$ and $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) = (\mathfrak{z}.\mathcal{M}x\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathfrak{V}(m) = U$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{z}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$. Since for all $(m.d) \in \mathcal{M}x\mathcal{D}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{z}(m.d) = \mho'(m) \cup \mathfrak{F}(d) \subseteq U = \mathfrak{V}(m)$, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{M}}$ is obtained. Similarly, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{D}}$ can be shown.

Proposition 21. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $(\mho.\mathcal{M})V_{+}(\mathfrak{F}.\mathcal{D}) =_{M} U_{\mathcal{M}x\mathcal{D}}$ if and only if $(\mho.\mathcal{M}) =_{M} \emptyset_{\mathcal{M}}$ and $(\mathfrak{F}.\mathcal{D}) =_{M} U_{\mathcal{D}}$.

Proof: let $U_{\mathcal{M}x\mathcal{D}} = (\mho.\mathcal{M}x\mathcal{D})$ and $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) = (\mathcal{V}.\mathcal{M}x\mathcal{D})$. Then, for all $(m.d) \in \mathcal{M}x\mathcal{D}$. $\mho(m.d) = U$ and for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$, Let $(\mho.\mathcal{M}x\mathcal{D}) = (\mathcal{V}.\mathcal{M}x\mathcal{D})$. Then, for all $(m.d) \in \mathcal{M}x\mathcal{D}$. $\mho'(m) \cup \mathfrak{F}(d) = U$, Thus, for all $m \in \mathcal{M}$. $\mho'(m) = U$ and for all $d \in \mathcal{D}$. $\mathfrak{F}(d) = U$. Thereby, $(\mho.\mathcal{M}) = \emptyset_{\mathcal{M}}$ and $(\mathfrak{F}.\mathcal{D}) = U_{\mathcal{D}}$.

Conversely, let $(\mho.\mathcal{M}) =_M \emptyset_{\mathcal{M}}$ and $(\mathfrak{F}.\mathcal{D}) =_M U_{\mathcal{D}}$. Then, for all $m \in \mathcal{M}$, $\mho(m) = \emptyset$ and for all $d \in \mathcal{D}$, $\mathfrak{F}(d) = U$. Thus, for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m.d) = \mho'(m) \cup \mathfrak{F}(d) = U \cup U = U$, implying that $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) =_M U_{\mathcal{M}x\mathcal{D}}$.

Proposition 22. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SSs over U. Then, $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$ if and only if $(\mho.\mathcal{M}) =_M \emptyset_\emptyset$ or $(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$,

Proof: let $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$, Then, $\mathcal{M}x\mathcal{D} = \emptyset$. and so $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$ Since $\emptyset_\emptyset$ is the only SS with the empty PS, $(\mho.\mathcal{M}) =_M \emptyset_\emptyset$ or $(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$ elde edilir.

Conversely, let $(\mho.\mathcal{M}) =_M \emptyset_\emptyset$ or $(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$, Thus, $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$, implying that $\mathcal{M}x\mathcal{D} = \emptyset$ and $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) =_M \emptyset_\emptyset$,

4 | Distribution of Soft Plus-Product Over Certain Types of Soft Set

In this section, we explore the distributions of soft plus-product over restricted, extended, soft binary piecewise intersection and union operations, and-product and or-product.

Theorem 1. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ and $(\mathcal{V}.\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft plus-product over restricted intersection and union operations:

I. $(\mho.\mathcal{M})V_+[(\mathfrak{F}.\mathcal{D}) \cup_R (\mathcal{V}.\mathcal{J})] =_M [(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D})] \cup_R [(\mho.\mathcal{M})V_+(\mathcal{V}.\mathcal{J})]$.

II. $(\mho.\mathcal{M})V_+[(\mathfrak{F}.\mathcal{D}) \cap_R (\mathcal{V}.\mathcal{J})] =_M [(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D})] \cap_R [(\mho.\mathcal{M})V_+(\mathcal{V}.\mathcal{J})]$.

III. $[(\mho.\mathcal{M}) \cap_R (\mathfrak{F}.\mathcal{D})]V_+(\mathcal{V}.\mathcal{J}) =_M [(\mho.\mathcal{M})V_+(\mathcal{V}.\mathcal{J})] \cup_R [(\mathfrak{F}.\mathcal{D})V_+(\mathcal{V}.\mathcal{J})]$.

IV. $[(\mho.\mathcal{M}) \cup_R (\mathfrak{F}.\mathcal{D})]V_+(\mathcal{V}.\mathcal{J}) =_M [(\mho.\mathcal{M})V_+(\mathcal{V}.\mathcal{J})] \cap_R [(\mathfrak{F}.\mathcal{D})V_+(\mathcal{V}.\mathcal{J})]$.

Proof:

I. The PS of the (left-hand side) LHS is $\mathcal{M}x(\mathcal{D} \cap \mathcal{J})$. The PS of the Right-Hand Side (RHS) is $(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J})$. Since $\mathcal{M}x(\mathcal{D} \cap \mathcal{J}) = (\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J})$, the first condition of the M-equality is satisfied. Let $(\mathfrak{F}.\mathcal{D}) \cup_R (\mathcal{V}.\mathcal{J}) = (\mathfrak{X}.\mathcal{D} \cap \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cap \mathcal{J}$, $\mathfrak{X}(\varphi) = \mathfrak{F}(\varphi) \cup \mathcal{V}(\varphi)$. Let $(\mho.\mathcal{M})V_+(\mathfrak{X}.\mathcal{D} \cap \mathcal{J}) = (\mathfrak{Z}.\mathcal{M}x(\mathcal{D} \cap \mathcal{J}))$, where for all $(m.\varphi) \in \mathcal{M}x(\mathcal{D} \cap \mathcal{J})$, $\mathfrak{Z}(m.\varphi) = \mho'(m) \cup \mathfrak{X}(\varphi)$. Then,

$$\mathfrak{Z}(m.\varphi) = \mho'(m) \cup [\mathfrak{F}(\varphi) \cup \mathcal{V}(\varphi)].$$

Suppose that $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) = (\mathfrak{T}.\mathcal{M}x\mathcal{D})$ and $(\mho.\mathcal{M})V_+(\mathcal{V}.\mathcal{J}) = (\wp.\mathcal{M}x\mathcal{J})$, where for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $\mathfrak{T}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$ and for all $(m.j) \in \mathcal{M}x\mathcal{J}$, $\wp(m.j) = \mho'(m) \cup \mathcal{V}(j)$. Let $(\mathfrak{T}.\mathcal{M}x\mathcal{D}) \cup_R (\wp.\mathcal{M}x\mathcal{J}) = (\mathfrak{Q}.(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}))$, where for all $(m.\varphi) \in (\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D} \cap \mathcal{J})$,

$\mathfrak{Q}(m.\varphi) = \mathfrak{T}(m.\varphi) \cup \wp(m.\varphi) = [\mho'(m) \cup \mathfrak{F}(d)] \cup [\mho'(m) \cup \mathcal{V}(j)]$.

Thus, $(\mho.\mathcal{M})V_+[(\mathfrak{F}.\mathcal{D}) \cup_R (\mathcal{V}.\mathcal{J})] =_M [(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D})] \cup_R [(\mho.\mathcal{M})V_+(\mathcal{V}.\mathcal{J})]$.

Here, if $\mathcal{D} \cap \mathcal{J} = \emptyset$, then $\mathcal{M}x(\mathcal{D} \cap \mathcal{J}) = (\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}) = \emptyset$. Since the only soft set with an empty PS is $\emptyset_\emptyset$, then both sides are $\emptyset_\emptyset$. Since $(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D} \cap \mathcal{J})$, if $(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J})=\emptyset$, then $\mathcal{M}=\emptyset$ or $\mathcal{D} \cap \mathcal{J} = \emptyset$. By assumption, $\mathcal{M} \neq \emptyset$. Thus, $(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J})=\emptyset$ implies that $\mathcal{D} \cap \mathcal{J} = \emptyset$. Therefore, under this condition, both sides are again $\emptyset_\emptyset$.

II. The PS of the LHS is $(\mathcal{M} \cap \mathcal{D})x\mathcal{J}$. and the PS of the RHS is $(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J})$, and since $(\mathcal{M} \cap \mathcal{D})x\mathcal{J} = (\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J})$, the first condition of M-equality is satisfied. Let $(\mho.\mathcal{M}) \cap_R (\mathfrak{F}.\mathcal{D}) = (\mathfrak{X}.\mathcal{M} \cap \mathcal{D})$, where for all $\varphi \in \mathcal{M} \cap \mathcal{D}$, $\mathfrak{X}(\varphi) = \mho(\varphi) \cap \mathfrak{F}(\varphi)$ and let $(\mathfrak{X}.\mathcal{M} \cap \mathcal{D})V_+(\mathcal{V}.\mathcal{J}) = (\mathfrak{Z}.(\mathcal{M} \cap \mathcal{D})x\mathcal{J})$, where for all $(\varphi.j) \in (\mathcal{M} \cap \mathcal{D})x\mathcal{J}$, $\mathfrak{Z}(\varphi.j) = \mathfrak{X}'(\varphi) \cup \mathcal{V}(j)$. Thereby,

$$ƻ(\varphi.j) = [\mho(\varphi) \cap \mathfrak{F}(\varphi)]' \cup \Upsilon(j) = [\mho'^{(\varphi)} \cup \mathfrak{F}'^{(\varphi)}] \cup \Upsilon(j).$$

Suppose that $(\mho.\mathcal{M})V_+(\Upsilon.\mathcal{J}) = (Ŧ.\mathcal{M}x\mathcal{J})$ and $(\mathfrak{F}.\mathcal{D})V_+(\Upsilon.\mathcal{J}) = (ღ.\mathcal{D}x\mathcal{J})$, where for all $(m.j) \in \mathcal{M}x\mathcal{J}$, $Ŧ(m.j) = \mho'(m) \cup \Upsilon(j)$ and $(d.j) \in \mathcal{D}x\mathcal{J}$, $ღ(d.j) = \mathfrak{F}'(d) \cup \Upsilon(j)$. Assume that $(Ŧ.\mathcal{M}x\mathcal{J}) \cup_R (ღ.\mathcal{D}x\mathcal{J}) = (☼.(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}))$, where for all $(\varphi.j) \in (\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})x\mathcal{J}$,

$$☼(\varphi.j) = Ŧ(\varphi.j) \cup ღ(\varphi.j) = [\mho'(\varphi) \cup \Upsilon(j)] \cup [\mathfrak{F}'(\varphi) \cup \Upsilon(j)].$$

Hence, $[(\mho.\mathcal{M}) \cap_R (\mathfrak{F}.\mathcal{D})]V_+(\Upsilon.\mathcal{J}) =_M [(\mho.\mathcal{M})V_+(\Upsilon.\mathcal{J})] \cup_R [(\mathfrak{F}.\mathcal{D})V_+(\Upsilon.\mathcal{J})]$.

Here, if $\mathcal{M} \cap \mathcal{D} = \emptyset$, then $(\mathcal{M} \cap \mathcal{D})x\mathcal{J} = (\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}) = \emptyset$. Since the only soft set with the empty parameter set is $\emptyset_\emptyset$, both sides of the equality is $\emptyset_\emptyset$. Moreover, since $(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})x\mathcal{J}$, if $(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J})=\emptyset$, then $\mathcal{M} \cap \mathcal{D} = \emptyset$ or $\mathcal{J}=\emptyset$. By assumption, $\mathcal{J} \neq \emptyset$. Thus, $(\mathcal{M}x\mathcal{J}) \cap (\mathcal{D}x\mathcal{J}) = \emptyset$ implies that $\mathcal{M} \cap \mathcal{D} = \emptyset$. Hence, under this condition, both sides of the equality are again $\emptyset_\emptyset$.

Note 1. The restricted soft set operation can not distribute over soft plus-product as the intersection does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 2. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ and $(\Upsilon.\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft plus-product over extended intersection and union operations:

I. $(\mho,\mathcal{M})V_+[(\mathfrak{F},\mathcal{D}) \cap_\varepsilon (\Upsilon,\mathcal{J})] =_M [(\mho,\mathcal{M})V_+(\mathfrak{F},\mathcal{D})] \cap_\varepsilon [(\mho,\mathcal{M})V_+(\Upsilon,\mathcal{J})]$.

II. $(\mho,\mathcal{M})V_+[(\mathfrak{F},\mathcal{D}) \cup_\varepsilon (\Upsilon,\mathcal{J})] =_M [(\mho,\mathcal{M})V_+(\mathfrak{F},\mathcal{D})] \cup_\varepsilon [(\mho,\mathcal{M})V_+(\Upsilon,\mathcal{J})]$.

III. $[(\mho,\mathcal{M}) \cup_\varepsilon (\mathfrak{F},\mathcal{D})]V_+(\Upsilon,\mathcal{J}) =_M [(\mho,\mathcal{M})V_+(\Upsilon,\mathcal{J})] \cap_\varepsilon [(\mathfrak{F},\mathcal{D})V_+(\Upsilon,\mathcal{J})]$.

IV. $[(\mho,\mathcal{M}) \cap_\varepsilon (\mathfrak{F},\mathcal{D})]V_+(\Upsilon,\mathcal{J}) =_M [(\mho,\mathcal{M})V_+(\Upsilon,\mathcal{J})] \cup_\varepsilon [(\mathfrak{F},\mathcal{D})V_+(\Upsilon,\mathcal{J})]$.

Proof:

I. The PS of the LHS is $\mathcal{M}x(\mathcal{D} \cup \mathcal{J})$. The PS of the RHS is $(\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J})$. Since $\mathcal{M}x(\mathcal{D} \cup \mathcal{J}) = (\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J})$, the first condition of the M-equality is satisfied. As $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$, $\mathcal{M}x(\mathcal{D} \cup \mathcal{J}) \neq \emptyset$ and $(\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J}) \neq \emptyset$. Thus, no side may be equal to an empty soft set. Let $(\mathfrak{F}.\mathcal{D}) \cap_\varepsilon (\Upsilon.\mathcal{J}) = (\mathfrak{X}.\mathcal{D} \cup \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cup \mathcal{J}$,

$$\mathfrak{X}(\varphi)=\begin{cases}\mathfrak{F}(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{J},\\ \Upsilon(\varphi), & \varphi\in\mathcal{J}\text{-}\mathcal{D},\\ \mathfrak{F}(\varphi) \cap \Upsilon(\varphi), & \varphi\in\mathcal{D} \cap \mathcal{J}.\end{cases}$$

Let $(\mho.\mathcal{M})V_+(\mathfrak{X}.\mathcal{D} \cup \mathcal{J}) = (ƻ.\mathcal{M}x(\mathcal{D} \cup \mathcal{J}))$, where for all $(m.\varphi) \in \mathcal{M}x(\mathcal{D} \cup \mathcal{J})$ için $ƻ(m.\varphi) =$

$\mho'(m) \cup \mathfrak{X}(\varphi)$. Thus, for all $(m.\varphi) \in \mathcal{M}x(\mathcal{D} \cup \mathcal{J})$,

$$ƻ(m.\varphi)=\begin{cases}\mho'(m) \cup \mathfrak{F}(\varphi), & (m.\varphi)\in\mathcal{M}x(\mathcal{D}\text{-}\mathcal{J}),\\ \mho'(m) \cup \Upsilon(\varphi), & (m.\varphi)\in\mathcal{M}x(\mathcal{J}\text{-}\mathcal{D}),\\ \mho'(m) \cup [\mathfrak{F}(\varphi) \cap \Upsilon(\varphi)], & (m.\varphi)\in\mathcal{M}x(\mathcal{D} \cap \mathcal{J}).\end{cases}$$

Suppose that $(\mho.\mathcal{M})V_+(\mathfrak{F}.\mathcal{D}) = (Ŧ.\mathcal{M}x\mathcal{D})$ and $(\mho.\mathcal{M})V_+(\Upsilon.\mathcal{J}) = (ღ.\mathcal{M}x\mathcal{J})$, where for all $(m.d) \in \mathcal{M}x\mathcal{D}$, $Ŧ(m.d) = \mho'(m) \cup \mathfrak{F}(d)$ and for all $(m.j) \in \mathcal{M}x\mathcal{J}$, $ღ(m.j) = \mho'(m) \cup \Upsilon(j)$. Assume that $(Ŧ.\mathcal{M}x\mathcal{D}) \cap_\varepsilon (ღ.\mathcal{M}x\mathcal{J}) = (☼.(\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J}))$, where for all $(m.\varphi) \in (\mathcal{M}x\mathcal{D}) \cup (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D} \cup \mathcal{J})$,

$$☼(m.\varphi)=\begin{cases}Ŧ(m.\varphi), & (m.\varphi)\in(\mathcal{M}x\mathcal{D})\text{-}(\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D}\text{-}\mathcal{J}),\\ ღ(m.\varphi), & (m.\varphi)\in(\mathcal{M}x\mathcal{J})\text{-}(\mathcal{M}x\mathcal{D}) = \mathcal{M}x(\mathcal{J}\text{-}\mathcal{D}),\\ Ŧ(m.\varphi) \cap ღ(m.\varphi), & (m.\varphi)\in(\mathcal{M}x\mathcal{D}) \cap (\mathcal{M}x\mathcal{J}) = \mathcal{M}x(\mathcal{D}\cap\mathcal{J}).\end{cases}$$

Hence,

$$\text{✡}(m,\varphi)=\begin{cases}\mho'(m)\cup\mathfrak{F}(\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J}),\\ \mho'(m)\cup\mathcal{V}(\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{D})=\mathcal{M}\text{x}(\mathcal{J}\text{-}\mathcal{D}),\\ [\mho'(m)\cup\mathfrak{F}(\varphi)]\cap[\mho'(m)\cup\mathcal{V}(\varphi)], & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D})\cap(\mathcal{M}\text{x}\mathcal{J})=\mathcal{M}\text{x}(\mathcal{D}\cap\mathcal{J}).\end{cases}$$

Thus, $(\mho,\mathcal{M})V_+[(\mathfrak{F},\mathcal{D})\cap_\varepsilon(\mathcal{V},\mathcal{J})]=_M[(\mho,\mathcal{M})V_+(\mathfrak{F},\mathcal{D})]\cap_\varepsilon[(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})]$.

II. The PS of the LHS is $(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J}$. and the PS of the RHS is $(\mathcal{M}\text{x}\mathcal{J})\cup(\mathcal{D}\text{x}\mathcal{J})$. Since $(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J}=(\mathcal{M}\text{x}\mathcal{J})\cup(\mathcal{D}\text{x}\mathcal{J})$, the first condition of the M-equality is satisfied. By assumption, $\mathcal{M}\neq\emptyset$, $\mathcal{D}\neq\emptyset$ and $\mathcal{J}\neq\emptyset$. Thus, $(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J}\neq\emptyset$ and $(\mathcal{M}\text{x}\mathcal{J})\cup(\mathcal{D}\text{x}\mathcal{J})\neq\emptyset$. Thereby, it is impossible that any of the sides is equal to an empty soft set. Let $(\mho,\mathcal{M})\cup_\varepsilon(\mathfrak{F},\mathcal{D})=(\mathfrak{X},\mathcal{M}\cup\mathcal{D})$, where for all $\varphi\in\mathcal{M}\cup\mathcal{D}$,

$$\mathfrak{X}(\varphi)=\begin{cases}\mho(\varphi), & \varphi\in\mathcal{M}\text{-}\mathcal{D},\\ \mathfrak{F}(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{M},\\ \mho(\varphi)\cup\mathfrak{F}(\varphi), & \varphi\in\mathcal{M}\cap\mathcal{D}.\end{cases}$$

Let $(\mathfrak{X},\mathcal{M}\cup\mathcal{D})V_+(\mathcal{V},\mathcal{J})=(\mathcal{Z},(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J})$, where for all $(\varphi,j)\in(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J}$, $\mathcal{Z}(\varphi,j)=\mathfrak{X}'(\varphi)\cup\mathcal{V}(j)$. Hence, for all $(\varphi,j)\in(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J}$,

$$\mathcal{Z}(\varphi,j)=\begin{cases}\mho'(\varphi)\cup\mathcal{V}(j), & (\varphi,j)\in(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J},\\ \mathfrak{F}'(\varphi)\cup\mathcal{V}(j), & (\varphi,j)\in(\mathcal{D}\text{-}\mathcal{M})\text{x}\mathcal{J},\\ [\mho'(\varphi)\cap\mathfrak{F}'(\varphi)]\cup\mathcal{V}(j), & (\varphi,j)\in(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}.\end{cases}$$

Assume that $(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})=(\mathcal{T},\mathcal{M}\text{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})V_+(\mathcal{V},\mathcal{J})=(\mathcal{L},\mathcal{D}\text{x}\mathcal{J})$, where for all $(m,j)\in\mathcal{M}\text{x}\mathcal{J}$, $\mathcal{T}(m,j)=\mho'(m)\cup\mathcal{V}(j)$ and for all $(d,j)\in\mathcal{D}\text{x}\mathcal{J}$, $\mathcal{L}(d,j)=\mathfrak{F}'(d)\cup\mathcal{V}(j)$. Suppose that $(\mathcal{T},\mathcal{M}\text{x}\mathcal{J})\cap_\varepsilon(\mathcal{L},\mathcal{D}\text{x}\mathcal{J})=(\text{✡},(\mathcal{M}\text{x}\mathcal{J})\cup(\mathcal{D}\text{x}\mathcal{J}))$, where for all $(\varphi,j)\in(\mathcal{M}\text{x}\mathcal{J})\cup(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\cup\mathcal{D})\text{x}\mathcal{J}$,

$$\text{✡}(\varphi,j)=\begin{cases}\mathcal{T}(\varphi,j), & (\varphi,j)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J},\\ \mathcal{L}(\varphi,j), & (\varphi,j)\in(\mathcal{D}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{J})=(\mathcal{D}\text{-}\mathcal{M})\text{x}\mathcal{J},\\ \mathcal{T}(\varphi,j)\cap\mathcal{L}(\varphi,j), & (\varphi,j)\in(\mathcal{M}\text{x}\mathcal{J})\cap(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}.\end{cases}$$

Thus,

$$\text{✡}(\varphi,j)=\begin{cases}\mho'(\varphi)\cup\mathcal{V}(j), & (\varphi,j)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J},\\ \mathfrak{F}'(\varphi)\cup\mathcal{V}(j), & (\varphi,j)\in(\mathcal{D}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{J})=(\mathcal{D}\text{-}\mathcal{M})\text{x}\mathcal{J},\\ [\mho'(\varphi)\cup\mathcal{V}(j)]\cap[\mathfrak{F}'(\varphi)\cup\mathcal{V}(j)], & (\varphi,j)\in(\mathcal{M}\text{x}\mathcal{J})\cap(\mathcal{D}\text{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\text{x}\mathcal{J}.\end{cases}$$

Hence, $[(\mho,\mathcal{M})\cup_\varepsilon(\mathfrak{F},\mathcal{D})]V_+(\mathcal{V},\mathcal{J})=_M[(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})]\cap_\varepsilon[(\mathfrak{F},\mathcal{D})V_+(\mathcal{V},\mathcal{J})]$.

Note 2. The extended soft set operation can not distribute over soft plus-product as the union operation does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

Theorem 3. Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over U. Then, we have the following distributions of soft plus-product over soft binary piecewise intersection and union operations:

I. $(\mho,\mathcal{M})V_+[(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\mathcal{V},\mathcal{J})]=_M[(\mho,\mathcal{M})V_+(\mathfrak{F},\mathcal{D})]\,\tilde{\cap}\,[(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})]$.

II. $(\mho,\mathcal{M})V_+[(\mathfrak{F},\mathcal{D})\,\tilde{\cup}\,(\mathcal{V},\mathcal{J})]=_M[(\mho,\mathcal{M})V_+(\mathfrak{F},\mathcal{D})]\,\tilde{\cup}\,[(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})]$.

III. $[(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})]V_+(\mathcal{V},\mathcal{J})=_M[(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})]\,\tilde{\cap}\,[(\mathfrak{F},\mathcal{D})V_+(\mathcal{V},\mathcal{J})]$.

IV. $[(\mho,\mathcal{M})\,\tilde{\cap}\,(\mathfrak{F},\mathcal{D})]V_+(\mathcal{V},\mathcal{J})=_M[(\mho,\mathcal{M})V_+(\mathcal{V},\mathcal{J})]\,\tilde{\cup}\,[(\mathfrak{F},\mathcal{D})V_+(\mathcal{V},\mathcal{J})]$.

Proof:

I. Since the PS of the SSs of both sides is $\mathcal{M}\text{x}\mathcal{D}$, the first condition of the M-equality is satisfied. Moreover, since $\mathcal{M}\neq\emptyset$ and $\mathcal{D}\neq\emptyset$ by assumption, $\mathcal{M}\text{x}\mathcal{D}\neq\emptyset$. Thus, it is impossible that any side is equal to an empty soft set. Let $(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(\mathcal{V},\mathcal{J})=(\mathfrak{X},\mathcal{D})$, where for all $d\in\mathcal{D}$,

$$\mathfrak{X}(d)=\begin{cases}\mathfrak{F}(d), & d\in\mathcal{D}\text{-}\mathcal{J},\\ \mathfrak{F}(d)\cap\mathcal{V}(d), & d\in\mathcal{D}\cap\mathcal{J}.\end{cases}$$

Let $(\mho.\mathcal{M})\vee_{+}(\mathfrak{X}.\mathcal{D})=(\wp.\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m.d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\wp(m.d)=\mho'(m)\cup\mathfrak{X}(d)$. Thus,

$$\wp(m.d)=\begin{cases}\mho'(m)\cup\mathfrak{F}(d), & (m.d)\in\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}),\\ \mho'(m)\cup[\mathfrak{F}(d)\cap\mathcal{V}(d)], & (m.d)\in\mathcal{M}\mathrm{x}(\mathcal{D}\cap\mathcal{J}).\end{cases}$$

Assume that $(\mho.\mathcal{M})\vee_{+}(\mathfrak{F}.\mathcal{D})=(\mathcal{T}.\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})=(\mathcal{Z}.\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m.d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{T}(m.d)=\mho'(m)\cup\mathfrak{F}(d)$ and for all $(m.\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{J}$, $\mathcal{Z}(m.\mathrm{j})=\mho'(m)\cup\mathcal{V}(\mathrm{j})$. Let $(\mathcal{T}.\mathcal{M}\mathrm{x}\mathcal{D})\,\tilde{\cap}\,(\mathcal{Z}.\mathcal{M}\mathrm{x}\mathcal{J})=(\mathcal{O}.\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m.d)\in\mathcal{M}\mathrm{x}\mathcal{D}$,

$$\mathcal{O}(m.d)=\begin{cases}\mathcal{T}(m.d), & (m.d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}),\\ \mathcal{T}(m.d)\cap\mathcal{Z}(m.d), & (m.d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\cap(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\cap\mathcal{J}).\end{cases}$$

Thus,

$$\mathcal{O}(m.d)=\begin{cases}\mho'(m)\cup\mathfrak{F}(d), & (m.d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J}),\\ [\mho'(m)\cup\mathfrak{F}(d)]\cap[\mho'(m)\cup\mathcal{V}(d)], & (m.d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\cap(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\cap\mathcal{J}).\end{cases}$$

Hence, $(\mho.\mathcal{M})\vee_{+}[(\mathfrak{F}.\mathcal{D})\,\tilde{\cap}\,(\mathcal{V}.\mathcal{J})]=_{\mathrm{M}}[(\mho.\mathcal{M})\vee_{+}(\mathfrak{F}.\mathcal{D})]\,\tilde{\cap}\,[(\mho.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})]$. Since $\mathcal{M}\neq\mathcal{M}\mathrm{x}\mathcal{M}$, the soft binary piecewise operations do not distribute over soft plus-product operations.

II. Since the PS of the SSs of both sides is $\mathcal{M}\mathrm{x}\mathcal{J}$, the first condition of the M-equality is satisfied. Moreover, since $\mathcal{M}\neq\emptyset$ and $\mathcal{J}\neq\emptyset$ by assumption, $\mathcal{M}\mathrm{x}\mathcal{J}\neq\emptyset$. Thus, it is impossible that any side is equal to an empty soft set. Let $(\mho.\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F}.\mathcal{D})=(\mathfrak{X}.\mathcal{M})$, where for all $m\in\mathcal{M}$,

$$\mathfrak{X}(m)=\begin{cases}\mho(m), & m\in\mathcal{M}\text{-}\mathcal{D},\\ \mho(m)\cup\mathfrak{F}(m). & m\in\mathcal{M}\cap\mathcal{D}.\end{cases}$$

Let $(\mathfrak{X}.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})=(\wp.\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m.\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{J}$, $\wp(m.\mathrm{j})=\mathfrak{X}'(m)\cup\mathcal{V}(\mathrm{j})$. Thus,

$$\mathfrak{X}(m.\mathrm{j})=\begin{cases}\mho'^{(m)}\cup\mathcal{V}(\mathrm{j}), & (m.\mathrm{j})\in(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J},\\ \left[\mho'^{(m)}\cap\mathfrak{F}'^{(m)}\right]\cup\mathcal{V}(\mathrm{j}), & (m.\mathrm{j})\in(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}.\end{cases}$$

Let $(\mho.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})=(\mathcal{T}.\mathcal{M}\mathrm{x}\mathcal{J})$ ve $(\mathfrak{F}.\mathcal{D})\vee_{+}(\mathcal{V}.\mathcal{J})=(\mathcal{Z}.\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m.\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{J}$, $\mathcal{T}(m.\mathrm{j})=\mho'(m)\cup\mathcal{V}(\mathrm{j})$ and for all $(d.\mathrm{j})\in\mathcal{D}\mathrm{x}\mathcal{J}$, $\mathcal{Z}(d.\mathrm{j})=\mathfrak{F}'(d)\cup\mathcal{V}(\mathrm{j})$ and let $(\mathcal{T}.\mathcal{M}\mathrm{x}\mathcal{J})\,\tilde{\cap}\,(\mathcal{Z}.\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{O}.\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m.\mathrm{j})\in\mathcal{M}\mathrm{x}\mathcal{J}$,

$$\mathcal{O}(m.\mathrm{j})=\begin{cases}\mathcal{T}(m.\mathrm{j}), & (m.\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J},\\ \mathcal{T}(m.\mathrm{j})\cap\mathcal{Z}(m.\mathrm{j}), & (m.\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\cap(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}.\end{cases}$$

There by,

$$\mathcal{O}(m.\mathrm{j})=\begin{cases}\mho'(m)\cup\mathcal{V}(\mathrm{j}), & (m.\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J},\\ [\mho'(m)\cup\mathcal{V}(\mathrm{j})]\cap[\mathfrak{F}'(m)\cup\mathcal{V}(\mathrm{j})], & (m.\mathrm{j})\in(\mathcal{M}\mathrm{x}\mathcal{J})\cap(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}.\end{cases}$$

Hence, $[(\mho.\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F}.\mathcal{D})]\vee_{+}(\mathcal{V}.\mathcal{J})=_{\mathrm{M}}[(\mho.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})]\,\tilde{\cap}\,[(\mathfrak{F}.\mathcal{D})\vee_{+}(\mathcal{V}.\mathcal{J})]$.

Proposition 23. Let $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ and $(\mathcal{V}.\mathcal{J})$ be SSs over U. Then,

I. $(\mho.\mathcal{M})\vee_{+}[(\mathfrak{F}.\mathcal{D})\wedge(\mathcal{V}.\mathcal{J})]\cong_{\mathrm{L}}[(\mho.\mathcal{M})\vee_{+}(\mathfrak{F}.\mathcal{D})]\wedge[(\mho.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})]$,

II. $(\mho.\mathcal{M})\vee_{+}[(\mathfrak{F}.\mathcal{D})\vee(\mathcal{V}.\mathcal{J})]\cong_{\mathrm{L}}[(\mho.\mathcal{M})\vee_{+}(\mathfrak{F}.\mathcal{D})]\vee[(\mho.\mathcal{M})\vee_{+}(\mathcal{V}.\mathcal{J})]$.

Proof:

I. Let $(\mathfrak{F}.\mathcal{D})\wedge(\mathcal{V}.\mathcal{J})=(\mho.\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(d.\mathrm{j})\in\mathcal{D}\mathrm{x}\mathcal{J}$, $\mho(d.\mathrm{j})=\mathfrak{F}(d)\cap\mathcal{V}(\mathrm{j})$ and $(\mho.\mathcal{M})\vee_{+}(\mho.\mathcal{D}\mathrm{x}\mathcal{J})=(\mathfrak{X}.\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(m.(d.\mathrm{j}))\in\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})$,

$$\text{Ӿ}(m.(d.\text{j})) = \mho'(m) \cup [\mathfrak{F}(d) \cap \text{℣}(\text{j})].$$

Assume that $(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D}) = (\text{☼}.\mathcal{M}\text{x}\mathcal{D})$ and $(\mho.\mathcal{M})\vee_+(\text{℣}.\mathcal{J}) = (\text{ƻ}.\mathcal{M}\text{x}\mathcal{J})$, where for all $(m.d) \in \mathcal{M}\text{x}\mathcal{D}$, $\text{☼}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$ and for all $(m.\text{j}) \in \mathcal{M}\text{x}\mathcal{J}$, $\text{ƻ}(m.\text{j}) = \mho'(m) \cup \text{℣}(\text{j})$. Let $(\text{☼}.\mathcal{M}\text{x}\mathcal{D})\Lambda(\text{ƻ}.\mathcal{M}\text{x}\mathcal{J}) = (\text{ღ}.(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J}))$, where for all $((m.d).(m.\text{j})) \in (\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})$,

$$\text{ღ}((m.d).(m.\text{j})) = [\mho'(m) \cup \mathfrak{F}(d)] \cap [\mho'(m) \cup \text{℣}(\text{j})].$$

Thus, for all $(m.(d.\text{j})) \in \mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})$, there exists $((m.d).(m.\text{j})) \in (\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})$ such that

$$\begin{aligned}\text{Ӿ}(m.(d.\text{j})) &= \mho'(m) \cup [\mathfrak{F}(d) \cap \text{℣}(\text{j})] = [\mho'(m) \cup \mathfrak{F}(d)] \cap [\mho'(m) \cup \text{℣}(\text{j})] \\ &= \text{ღ}((m.d).(m.\text{j})).\end{aligned}$$

This completes the proof. It is obvious that the L-subset in *Proposition 23*. can not be L-equality with the following example.

Example 4. Let $\text{E} = \{e_1.e_2.e_3.e_4.e_5\}$ be the parameter set, $\mathcal{M} = \{e_1.e_5\}$, $\mathcal{D} = \{e_3\}$ and $\mathcal{J} = \{e_2\}$ be the subsets of E, $\text{U} = \{h_1.h_2.h_3.h_4.h_5.h_6\}$ be the universal set and $(\mho.\mathcal{M})$, $(\mathfrak{F}.\mathcal{D})$ and $(\text{℣}.\mathcal{J})$ be SSs over U as follows: $(\mho.\mathcal{M}) = \{(e_1.\{h_1.h_6\}).(e_5.\{h_2.h_4.h_5\})\}$, $(\mathfrak{F}.\mathcal{D}) = \{(e_3.\{h_1.h_3.h_4\})\}$ and $(\text{℣}.\mathcal{J}) = \{(e_2.\{h_1.h_4.h_5\})\}$. We show that

$$(\mho.\mathcal{M})\vee_+[(\mathfrak{F}.\mathcal{D})\Lambda(\text{℣}.\mathcal{J})] \neq_{\text{L}} [(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})]\Lambda[(\mho.\mathcal{M})\vee_+(\text{℣}.\mathcal{J})].$$

Let $(\mathfrak{F}.\mathcal{D})\Lambda(\text{℣}.\mathcal{J}) = (\text{Ʊ}.\mathcal{D}\text{x}\mathcal{J})$, where

$$(\mathfrak{F}.\mathcal{D})\Lambda(\text{℣}.\mathcal{J}) = (\text{Ʊ}.\mathcal{D}\text{x}\mathcal{J}) = \{((e_3.e_2).\{h_1.h_4\})\}.$$

Suppose that $(\mho.\mathcal{M})\vee_+(\text{Ʊ}.\mathcal{D}\text{x}\mathcal{J}) = (\text{Ӿ}.\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J}))$, where

$$(\text{Ӿ}.\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})) = \Big\{\big((e_1.(e_3.e_2)).\{h_1.h_2.h_3.h_4.h_5\}\big).\big((e_5.(e_3.e_2)).\{h_1.h_3.h_4.h_6\}\big)\Big\}.$$

Assume that $(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D}) = (\text{ƻ}.\mathcal{M}\text{x}\mathcal{D})$, where

$$(\text{ƻ}.\mathcal{M}\text{x}\mathcal{D}) = \{((e_1.e_3).\{h_1.h_2.h_3.h_4.h_5\}).((e_5.e_3).\{h_1.h_3.h_4.h_6\})\}.$$

Suppose that $(\mho.\mathcal{M})\vee_+(\text{℣}.\mathcal{J}) = (\text{☼}.\mathcal{M}\text{x}\mathcal{J})$, where

$$(\text{☼}.\mathcal{M}\text{x}\mathcal{J}) = \{((e_1.e_2).\{h_1.h_2.h_3.h_4.h_5\}).((e_5.e_2).\{h_1.h_3.h_4.h_5.h_6\})\}.$$

Let $(\text{ƻ}.\mathcal{M}\text{x}\mathcal{D})\Lambda(\text{☼}.\mathcal{M}\text{x}\mathcal{J}) = (\text{ღ}.(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})$, Then,

$$\begin{aligned}&(\text{ღ}.(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})) \\ &= \{(((e_1.e_3).(e_1.e_2)).\{h_1.h_2.h_3.h_4.h_5\}).(((e_1.e_3).(e_5.e_2)).\{h_1.h_3.h_4.h_5\}). \\ &\quad (((e_5.e_3).(e_1.e_2)).\{h_1.h_3.h_4\}).(((e_5.e_3).(e_5.e_2)).\{h_1.h_3.h_4.h_6\})\}.\end{aligned}$$

Thereby, $\text{ღ}((e_1.e_3).(e_5.e_2)) \neq \text{Ӿ}(e_1.(e_3.e_2))$, $\text{ღ}((e_1.e_3).(e_5.e_2)) \neq \text{Ӿ}((e_5.(e_3.e_2))$, $\text{ღ}((e_5.e_3).(e_1.e_2)) \neq \text{Ӿ}(e_1.(e_3.e_2))$ and $\text{ღ}((e_5.e_3).(e_1.e_2)) \neq \text{Ӿ}((e_5.(e_3.e_2))$ implying that $(\text{ღ}.(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})) \nsubseteq_{\text{L}} (\text{Ӿ}.\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J}))$,

5 | İnt-Uni Decısıon-Makıng Method Applıed to Soft Plus-Product

In this section, the int-uni operator and int-uni decision function defined by Çağman and Enginoğlu [11] are applied for the soft plus-product in this part to create an int-uni decision-making method.

Throughout this section, all the soft plus-products ($\vee_+$) of the SSs over U are assumed to be contained in the set $\vee_+(U)$, and the approximation function of the soft plus-product of $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$, that is, $(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D})$ is

$$\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}: \mathcal{M}\mathrm{x}\mathcal{D} \to P(U),$$

where $\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}(m.d) = \mho'(m) \cup \mathfrak{F}(d)$ for all $(m.d) \in \mathcal{M}\mathrm{x}\mathcal{D}$.

Definition 15. Let $(\mho.\mathcal{M})$ and $(\mathfrak{F}.\mathcal{D})$ be SS over U. Then, int-uni operators for soft plus-product, denoted by $\mathrm{int}_x\mathrm{uni}_y$ and $\mathrm{int}_y\mathrm{uni}_x$ are defined respectively as

$$\mathrm{int}_x\mathrm{uni}_y: \vee_+ \to P(U), \qquad \mathrm{int}_x\mathrm{uni}_y: (\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}) = \bigcap_{m\in\mathcal{M}}(\bigcup_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}(m.d))),$$
$$\mathrm{int}_y\mathrm{uni}_x: \vee_+ \to P(U), \qquad \mathrm{int}_y\mathrm{uni}_x\,(\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}) = \bigcap_{d\in\mathcal{D}}(\bigcup_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}(m.d))).$$

Definition 16 ([11]). Let $(\mho.\mathcal{M})\vee_+(\mathfrak{F}.\mathcal{D}) \in \vee_+(U)$. Then, the int-uni decision function for soft difference-product, denoted by int-uni are defined by

$$\text{int-uni}: \vee_+ \to P(U), \qquad \text{int-uni}\,(\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}) = \mathrm{int}_x\mathrm{uni}_y\,(\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}) \cup \mathrm{int}_y\mathrm{uni}_x: (\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}).$$

The values int-uni $(\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}})$ is a subset of U called int-uni decision set of $\mho_{\mathcal{M}}\vee_+\mathfrak{F}_{\mathcal{D}}$.

Assume that a set of parameters and a set of options are provided. The int-uni decision-making method, which is structured as follows, is then used to choose a collection of optimal options while keeping the problem in mind.

Step 1. Select feasible subsets from the parameter collection,

Step 2. Build the SSs for every parameter set.

Step 3. Determine the SSs' soft plus product,

Step 4. Determine the produce int-uni decision set.

We are now ready to show how soft set theory may be used for the uni-int decision-making problem for the soft-plus product.

Example 5. The pilot recruitment process includes steps such as interviews, psychotechnical evaluations, simulator flights, and medical assessments. The candidate's English language proficiency, technical knowledge, mathematics and physics skills, visual memory, and social skills also play a significant role in the selection process. An airline company has announced a recruitment process to build a young and dynamic team of pilots. Applications are initially reviewed to ensure they meet the required qualifications for the position, and any application that does not meet the criteria is disqualified. Due to the high number of remaining applications after disqualifying the ineligible candidates, the company decided to implement a two-stage evaluation process. In this process, Mr. Utku, a representative from the company's Human Resources department, will first ELIMINATE pilot candidates based on interviews and exam results. The remaining candidates will then undergo a comprehensive training program, and those who successfully complete it will qualify to join the company's professional pilot team. While making his decisions, Mr Utku will consider the parameters he absolutely does not want to see in candidates to be ELIMINATED and the parameters he does wish to see in such candidates. His decision will be made using the soft plus-product int-uni decision-making method.

Let the set of candidates whose applications have been validated for the pilot recruitment process be:

$U = \{z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{11}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}.$

$z_{26}. z_{27}. z_{28}. z_{29}. z_{30}. z_{31}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40}\}.$

Let the set of parameters to be used for identifying the pilots to be eliminated be $\mathrm{E} = \{c_1. c_2. c_3. c_4. c_5. c_6. c_7. c_8\}$ Here, the parameters c_i, $\mathrm{i} = 1.2.3 \dots 8$ represent the following, respectively:

I. c_1: "Confident and self-disciplined".

II. c_2: "Insufficient mathematical knowledge and creative ability".

III. c_3: "Possesses leadership skills and the ability to work as a team".

IV. c_4: "Lacks situational awareness".

V. c_5: "Does not have open communication skills".

VI. c_6: "High ability to comprehend technical information".

VII. c_7: "Low ability to remain calm under stress".

VIII. c_8: "Proficient in English".

To solve this problem related to the pilot selection process, we can apply the soft plus-product method to the problem as follows:

Step 1. Determining the sets of parameters.

Decision-maker Mr. Utku selects parameters from the existing set that define the characteristics of candidates to be eliminated.

I. Parameters that must absolutely not be present in eliminated candidates: These represent traits or skills that are essential and desired in a pilot, and their absence disqualifies a candidate.

II. Parameters that are preferred to be present in candidates to be eliminated: These represent undesirable traits or deficiencies that make a candidate unsuitable for selection.

By categorizing these parameters into two sets, the selection process ensures clarity and alignment with the decision-makers' priorities. These parameter sets are $\mathcal{M} = \{c_1. c_3. c_6. c_8\}$ and $\mathcal{D} = \{c_2. c_4. c_5. c_7\}$, respectively.

Step 2. Constructing the SSs by using the PSs determined in *Step 1*.

Using these parameter sets, the decision-maker constructs the soft sets $(\mho, \mathcal{M})$ ve $(\mho, \mathcal{D})$:

$$(\mho. \mathcal{M}) = \{(c_1. \{z_2. z_6. z_7. z_8. z_{11}. z_{17}. z_{19}. z_{24}. z_{26}. z_{31}. z_{33}. z_{38}. z_{39}. z_{40}\}). (c_3. \{z_2. z_3. z_6. z_7. z_{11}.$$

$$z_{17}. z_{21}. z_{23}. z_{30}. z_{31}. z_{32}. z_{33}. z_{39}\}). (c_6. \{z_6. z_9. z_{11}. z_{16}. z_{18}. z_{19}. z_{21}. z_{22}. z_{24}. z_{30}. z_{31}. z_{32}. z_{33}\}).$$

$$\left(c_8. \{z_1. z_3. z_6. z_{10}. z_{11}. z_{13}. z_{17}. z_{22}. z_{23}. z_{25}. z_{28}. z_{29}. z_{30}. z_{31}. z_{38}. z_{39}. z_{40}\right\}\}.$$

$$(\mho. \mathcal{D}) = \{(c_2. \{z_8. z_9. z_{12}. z_{14}. z_{16}. z_{17}. z_{22}. z_{25}. z_{32}. z_{33}\}). (c_4. \{z_1. z_3. z_5. z_7. z_{12}. z_{15}. z_{19}. z_{20}. z_{21}. z_{35}. z_{36}. z_{37}\}).$$

$$(c_5. [z_3. z_4. z_6. z_9. z_{13}. z_{18}. z_{19}. z_{25}. z_{28}. z_{38}. z_{40}\}). (c_7. \{z_7. z_{10}. z_{13}. z_{14}. z_{19}. z_{20}. z_{23}. z_{27}. z_{29}. z_{30}. z_{34}. z_{37}. z_{40})\}\}.$$

$(\mho, \mathcal{M})$ is an SS representing candidates close to the ideal by possessing the highly desired parameters in $\mathcal{M}$, $(\mho, \mathcal{D})$ is an SS representing candidates to be eliminated due to undesirable parameters in $\mathcal{D}$. The process of constructing these sets involves weighting and evaluating the importance of each parameter in decision-making, ensuring a balanced and justifiable elimination process. Note that Mr Utku's task is to ELIMINATE the candidates.

Step 3. Determine the $\vee_{+}$-product of soft sets:

$\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}} =$

$$\{\left((c_1.c_2).\left\{\begin{matrix} z_1.z_3.z_4.z_5.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}. \\ z_{27}.z_{30}.z_{32}.z_{33}.z_{34}.z_{35}.z_{36}.z_{37} \end{matrix}\right\}\right).$$

$$\left((c_1.c_4).\left\{\begin{matrix} z_1.z_3.z_4.z_5.z_7.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}. \\ z_{27}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37} \end{matrix}\right\}\right).$$

$$\left((c_1.c_5).\left\{\begin{matrix} z_1.z_3.z_4.z_5.z_6.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}. \\ z_{27}.z_{28}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_1.c_7).\left\{\begin{matrix} z_1.z_3.z_4.z_5.z_7.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}. \\ z_{27}.z_{29}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_3.c_2).\left\{\begin{matrix} z_1.z_4.z_5.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}. \\ z_{28}.z_{29}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_3.c_4).\left\{\begin{matrix} z_1.z_3.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}. \\ z_{28}.z_{29}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_3.c_5).\left\{\begin{matrix} z_1.z_3.z_4.z_5.z_6.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}. \\ z_{28}.z_{29}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_3.c_7).\left\{\begin{matrix} z_1.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{22}.z_{23}.z_{24}.z_{25}.z_{26}.z_{27}. \\ z_{28}.z_{29}.z_{30}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_6.c_2).\left\{\begin{matrix} z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{20}.z_{22}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}. \\ z_{32}.z_{33}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_6.c_4).\left\{\begin{matrix} z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{17}.z_{19}.z_{20}.z_{21}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}. \\ z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_6.c_5).\left\{\begin{matrix} z_1.z_2.z_3.z_4.z_5.z_6.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{17}.z_{18}.z_{19}.z_{20}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}. \\ z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_6.c_7).\left\{\begin{matrix} z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{17}.z_{19}.z_{20}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}. \\ z_{30}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_8.c_2).\left\{\begin{matrix} z_2.z_4.z_5.z_7.z_8.z_9.z_{12}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}.z_{29}.z_{32}. \\ z_{33}.z_{34}.z_{35}.z_{36}.z_{37} \end{matrix}\right\}\right).$$

$$\left((c_8.c_4).\left\{\begin{matrix} z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_9.z_{12}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{24}.z_{26}.z_{27}.z_{29}.z_{32}. \\ z_{33}.z_{34}.z_{35}.z_{36}.z_{37} \end{matrix}\right\}\right).$$

$$\left((c_8.c_5).\left\{\begin{matrix} z_2.z_3.z_4.z_5.z_6.z_7.z_8.z_9.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{24}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}.z_{32}. \\ z_{33}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40} \end{matrix}\right\}\right).$$

$$\left((c_8.c_7).\left\{\begin{matrix} z_2.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{23}.z_{24}.z_{26}.z_{27}.z_{29}.z_{30}.z_{32}. \\ z_{33}.z_{34}.z_{35}.z_{36}.z_{37}.z_{40} \end{matrix}\right\}\right),$$

Step 4. Determine the set of int-uni($\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}}$):

$$\mathrm{int}_m-\mathrm{uni}_d(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})=\cap_{m\in\mathcal{M}}\left(\cup_{d\in\mathcal{D}}\left((\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(m.d)\right)\right).$$

We first determine $\cup_{d\in\mathcal{D}}\left((\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(m,d)\right)$:

$(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_1.c_2)\cup\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_1.c_4)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_1.c_5)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_1.c_7)$

$$=\begin{Bmatrix}z_1.z_3.z_4.z_5.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}.\\ z_{27}.z_{30}.z_{32}.z_{33}.z_{34}.z_{35}.z_{36}.z_{37}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_3.z_4.z_5.z_7.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}.\\ z_{27}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_3.z_4.z_5.z_6.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}.\\ z_{27}.z_{28}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{40}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_3.z_4.z_5.z_7.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}.\\ z_{27}.z_{29}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{40}\end{Bmatrix}$$

$$=\begin{Bmatrix}z_1.z_3.z_4.z_5.z_6.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}.\\ z_{27}.z_{28}.z_{29}.z_{30}.z_{32}.z_{33}.z_{34}.z_{35}.z_{36}.z_{37}.z_{40}\end{Bmatrix}.$$

$(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_3.c_2)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_3.c_4)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_3.c_5)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_3.c_7)$

$$=\begin{Bmatrix}z_1.z_4.z_5.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}.\\ z_{28}.z_{29}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_3.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}.\\ z_{28}.z_{29}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40}\end{Bmatrix},$$

$$\cup\begin{Bmatrix}z_1.z_3.z_4.z_5.z_6.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}.\\ z_{28}.z_{29}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{18}.z_{19}.z_{20}.z_{22}.z_{23}.z_{24}.z_{25}.z_{26}.z_{27}.\\ z_{28}.z_{29}.z_{30}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40}\end{Bmatrix}$$

$$=\begin{Bmatrix}z_1.z_3.z_4.z_5.z_6.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{24}.z_{25}.z_{26}.z_{27}.\\ z_{28}.z_{29}.z_{30}.z_{32}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{40}\end{Bmatrix}.$$

$(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_6.c_2)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_6.c_4)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_6.c_7)$

$$=\begin{Bmatrix}z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{20}.z_{22}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}.\\ z_{32}.z_{33}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{17}.z_{19}.z_{20}.z_{21}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}.\\ z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_2.z_3.z_4.z_5.z_6.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{17}.z_{18}.z_{19}.z_{20}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}.\\ z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40}\end{Bmatrix}.$$

$$\cup\begin{Bmatrix}z_1.z_2.z_3.z_4.z_5.z_7.z_8.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{17}.z_{19}.z_{20}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}.\\ z_{30}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40}\end{Bmatrix}$$

$$=\begin{Bmatrix}z_1.z_2.z_3.z_4.z_5.z_6.z_7.z_8.z_9.z_{10}.z_{12}.z_{13}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{23}.z_{25}.z_{26}.z_{27}.z_{28}.z_{29}.\\ z_{30}.z_{32}.z_{33}.z_{34}.z_{35}.z_{36}.z_{37}.z_{38}.z_{39}.z_{40}\end{Bmatrix}.$$

$(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_8.c_2)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_8.c_4)\cup(\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_8.c_5)\cup\mho_{\mathcal{M}}\mathrm{V}_{+}\mho_{\mathcal{D}})(c_8.c_7)$

$$=\begin{Bmatrix}z_2.z_4.z_5.z_7.z_8.z_9.z_{12}.z_{14}.z_{15}.z_{16}.z_{17}.z_{18}.z_{19}.z_{20}.z_{21}.z_{22}.z_{24}.z_{25}.z_{26}.z_{27}.z_{29}.z_{32}.\\ z_{33}.z_{34}.z_{35}.z_{36}.z_{37}\end{Bmatrix}.$$

$$\cup \left\{\begin{matrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{12}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{24}. z_{26}. z_{27}. z_{29}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{24}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} z_2. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{23}. z_{24}. z_{26}. z_{27}. z_{29}. z_{30}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{40} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. \\ z_{27}. z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{matrix}\right\}.$$

Thus,

$$int_m - uni_d(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}}) = \cap_{m\in\mathcal{M}} \left(\cup_{d\in\mathcal{D}}\left(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}}(m.d)\right)\right) =$$

$$\left\{\begin{matrix} z_1. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. \\ z_{27}. z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{40} \end{matrix}\right\},$$

$$\cap \left\{\begin{matrix} z_1. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{matrix}\right\}.$$

$$\cap \left\{\begin{matrix} z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{matrix}\right\}.$$

$$\cap \left\{\begin{matrix} z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. \\ z_{27}. z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{matrix}\right\}$$

$$= \{z_1. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. z_{27}. z_{28}.$$

$$z_{29}. z_{30}. z_{32}. z_{34}. z_{35}. z_{36}. z_{37} z_{40}\},$$

is obtained.

$$\mathrm{int}_d - \mathrm{uni}_m(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}}) = \cap_{d\in\mathcal{D}} \left(\cup_{m\in\mathcal{M}} \left((\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(m.d)\right)\right).$$

We first determine $\cup_{m\in\mathcal{M}} \left((\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(m,d)\right)$:

$$(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_1. c_2) \cup (\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_3. c_2) \cup (\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_6. c_2) \cup (\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_8. c_2)$$

$$= \left\{\begin{matrix} z_1. z_3. z_4. z_5. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. \\ z_{27}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} z_1. z_4. z_5. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{22}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{32}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{20}. z_{22}. z_{23}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{matrix}\right\},$$

$$\cup \left\{\begin{matrix} z_2. z_4. z_5. z_7. z_8. z_9. z_{12}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{24}. z_{25}. z_{26}. z_{27}. z_{29}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{matrix}\right\},$$

$$(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_1. c_4) \cup (\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_3. c_4) \cup (\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_6. c_4) \cup (\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})(c_8. c_4)$$

$$= \left\{\begin{matrix} z_1. z_3. z_4. z_5. z_7. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. \\ z_{27}. z_{30}. z_{32}. z_{34}. z_{35}. z_{36}. z_{37} \end{matrix}\right\}.$$

$$\cup\begin{Bmatrix} z_1. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{17}. z_{19}. z_{20}. z_{21}. z_{23}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{12}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{24}. z_{26}. z_{27}. z_{29}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37} \end{Bmatrix}$$

$$=\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$(\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_1. c_5) \cup (\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_3. c_5) \cup (\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_6. c_5) \cup (\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_8. c_5)$$

$$=\begin{Bmatrix} z_1. z_3. z_4. z_5. z_6. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. \\ z_{27}. z_{28}. z_{30}. z_{32}. z_{34}. z_{35}. z_{36}. z_{37}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_1. z_3. z_4. z_5. z_6. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{22}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{17}. z_{18}. z_{19}. z_{20}. z_{23}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{24}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{Bmatrix}$$

$$=\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$(\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_1. c_7) \cup (\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_3. c_7) \cup (\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_6. c_7) \cup (\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(c_8. c_7)$$

$$=\begin{Bmatrix} z_1. z_3. z_4. z_5. z_7. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. \\ z_{27}. z_{29}. z_{30}. z_{32}. z_{34}. z_{35}. z_{36}. z_{37}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_1. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{40} \end{Bmatrix},$$

$$\cup\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{17}. z_{19}. z_{20}. z_{23}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{30}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$\cup\begin{Bmatrix} z_2. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{18}. z_{19}. z_{20}. z_{21}. z_{23}. z_{24}. z_{26}. z_{27}. z_{29}. z_{30}. z_{32}. \\ z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{40} \end{Bmatrix}$$

$$=\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix},$$

is obtained. Therefore,

$$\mathrm{int}_d - \mathrm{uni}_m(\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}}) = \cap_{d\in\mathcal{D}}\left(\cup_{m\in\mathcal{M}}\left((\mho_{\mathcal{M}} \vee_{+} \mho_{\mathcal{D}})(m. d)\right)\right)$$

$$\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. z_{28}. z_{29}. \\ z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$\cap\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$\cap\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}.$$

$$\cap\begin{Bmatrix} z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{26}. z_{27}. \\ z_{28}. z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40} \end{Bmatrix}$$

$= \{z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{27}. z_{28}.$

$z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40}\}.$

Thus,

$\text{int-uni}(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}}) = [\text{int}_{m} - \text{uni}_{d}(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})] \cup [\text{int}_{d} - \text{uni}_{m}(\mho_{\mathcal{M}} \mathrm{V}_{+} \mho_{\mathcal{D}})]$

$= \{z_1. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{25}. z_{27}. z_{28}. z_{29}. z_{30}.$

$z_{32}. z_{34}. z_{35}. z_{36}. z_{37} z_{40}\}.$

$\cup \{z_1. z_2. z_3. z_4. z_5. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{27}. z_{28}. z_{29}. z_{30}.$

$z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40}\}$

$= \{z_1. z_2. z_3. z_4. z_5. z_6. z_7. z_8. z_9. z_{10}. z_{12}. z_{13}. z_{14}. z_{15}. z_{16}. z_{17}. z_{18}. z_{19}. z_{20}. z_{21}. z_{22}. z_{23}. z_{24}. z_{25}. z_{27}. z_{28}.$

$z_{29}. z_{30}. z_{32}. z_{33}. z_{34}. z_{35}. z_{36}. z_{37}. z_{38}. z_{39}. z_{40}\}.$

Thus, in the airline company's pilot recruitment process, out of the 40 candidates whose applications were accepted, 37 were eliminated in the first stage. The remaining candidates $\{z_{11}, z_{26}, z_{31}\}$ were included in a comprehensive training program and earned the right to join the company's professional pilot team.

6 | Conclusion

In this study, we first introduced a novel product for soft sets, which we call the "Soft plus-product", utilizing Molodtsov's [2] soft set. With regard to a number of soft subset and soft equality types, such as M-subset/equality, L-subset/equality, and J-subset/equality, we carefully examined its algebraic aspects. Additionally, the distibutions of soft plus-product over different types of soft set operations were acquired. Finally, we applied a soft decision-making technique that chooses the best components from possibilities without using fuzzy soft sets or rough sets. We also provided an example that demonstrates how the technique may be effectively applied in a range of areas. Numerous applications, including innovative soft set-based cryptography methods and novel approaches to decision-making, are possible by this study. To expand on this work and add to the soft set literature from a theoretical and practical perspective, future research may suggest some additional soft product operations and further study basic features related to various types of soft equal relations.

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Data Availability

All data are included in the text.

Conflict of Interest

The authors declare no conflict of interest.

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