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Quadropoly Game with Heterogeneous Players

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Abstract

This paper explores the heterogeneous quadropoly game, aiming to examine the stability conditions of equilibrium points and the emergence of complex dynamics. Four firms with distinct strategies—two naive, one adaptive, and one bounded rational—are modeled within a linear discrete-time dynamical system. Using the Jacobian matrix and Schur-Cohn criterion, we find boundary equilibrium points to be locally unstable, while Nash equilibrium stability depends on the adaptive firm's adjustment rate. This research extends traditional oligopoly models, providing insights into strategic decision-making and profit maximization in complex oligopoly markets.

Keywords: Bounded rational, Adaptive strategy, Naive, Equilibrium, Discrete Dynamical System, Jacobian matrix, Schur Cohn Criterion.

1|Introduction

Oligopoly is amonst the oldest discipline of mathematical economics. The first mathematical model of oligopoly was proposed by Cournot[1].Game theory is the eminent perspective to review oligopoly. Oligopoly is a system where few firms rule the market and make strategis to earn the profit[2].Many researchers have considered different aspects related to Oligopoly [3, 4, 5, 6]. There are many studies about the stability analysis in duopoly and triopoly [7, 8, 9, 10]with homogeneous expectations. Impact of heterogeneous expectations on equilibrium has also been topic of interest for many scholars[11, 12, 13, 14].There are eminent studies which draw own

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attention on complex dynamics of non-linear models in oligopoly [15, 16, 17, 18].Quadropoly is that part of oligopoly market system, where four firms are involved in profit maximization venture. An economic system becomes more realistic when the number of firms in the system is large. A dynamical system, in which more players are involved, is more complicated and more interesting. In this paper, a linear discrete-time dynamical Quadropoly model with heterogeneous strategies are formulated and conditions for the stability of fixed points are discussed.

In the underlying model of Quadropoly, the underlying assumption is that first and second firms are naive, third firm is adaptive and fourth one is bounded rational.

2|Quadropoly Model

Let there be four firms and $x_i(t)$, $i = 1, 2, 3, 4$ be output produced by i^{th} firm at any time t. Inverse demand function is $Y = a - bX$.
$X = x_1(t) + x_2(t) + x_3(t) + x_4(t)$ represents total supply of the market.Let c_i ,$i = 1, 2, 3, 4$ be the marginal cost of production of the i^{th} firm. Profit function of the i^{th} firm is
$\pi_i = x_i(a - bX) - c_i x_i$,$i = 1, 2, 3, 4$
Here x_i is written in place of $x_i(t)$ for the sake of simplicity.
Marginal profits of all the four firms are:
$\frac{\partial \pi_1}{\partial x_1} = a - 2bx_1 - bx_2 - bx_3 - bx_4 - c_1$
$\frac{\partial \pi_2}{\partial x_2} = a - bx_1 - 2bx_2 - bx_3 - bx_4 - c_2$
$\frac{\partial \pi_3}{\partial x_3} = a - bx_1 - bx_2 - 2bx_3 - bx_4 - c_3$
$\frac{\partial \pi_4}{\partial x_4} = a - bx_1 - bx_2 - bx_3 - 2bx_4 - c_4$
Each firm wants to maximize respective profit. So value of output x_i of the i^{th} firm is found, for which its marginal profit $\frac{\partial \pi_i}{\partial x_i} = 0$, $i = 1, 2, 3, 4$
For

$$\frac{\partial \pi_1}{\partial x_1} = 0, x_1 = \frac{a - c_1 - b(x_2 + x_3 + x_4)}{2b} \tag{1}$$

Similarly,

$$x_2 = \frac{a - c_2 - b(x_1 + x_3 + x_4)}{2b} \tag{2}$$

$$x_3 = \frac{a - c_3 - b(x_1 + x_2 + x_4)}{2b} \tag{3}$$

$$x_4 = \frac{a - c_4 - b(x_1 + x_2 + x_3)}{2b} \tag{4}$$

These are called reaction functions. Using the concept of maxima minima, it is found that for these values of output, profit is the maximum. So, these are profit maximization level of output. So, Dynamical equations of the first two firms with naive expectations are as under:

$$x_1(t+1) = \frac{a - c_1 - b(x_2(t) + x_3(t) + x_4(t))}{2b} \tag{5}$$

$$x_2(t+1) = \frac{a - c_2 - b(x_1(t) + x_3(t) + x_4(t))}{2b} \tag{6}$$

Third firm being adaptive, calculates level of output to be produced using weighted average of reaction function given in equation (2.3) and level of previous output at time t .i.e. $x_3(t)$. Dynamical equation of the third firm is

$$x_3(t+1) = (1 - \alpha)x_3(t) + \alpha \left(\frac{a - c_3 - b(x_1(t) + x_2(t) + x_4(t))}{2b} \right) \tag{7}$$

where $0 \leq \alpha \leq 1$ indicates rate with which the adaptive firm adjusts output with respect to market conditions. Fourth firm occurs to be bounded rational. Dynamical equation of bounded rational firm is

$$x_4(t+1) = x_4(t) + \beta x_4(t)[a - 2bx_4(t) - b(x_1(t) + x_2(t) + x_3(t)) - c_4] \tag{8}$$

3|Calculating Boundary and Nash Equilibrium Points with Stability Conditions

For finding Boundary and Nash Equilibrium points of the linear Quadropoly game, it is required to determine the non-negative fixed point of the system of nonlinear equations (2.5), (2.6), (2.7) and (2.8). So, taking $x_i(t+1) = x_i(t)$, $i = 1, 2, 3, 4$ in each of (2.5), (2.6), (2.7) and (2.8) the system of equations is as under:

$$\frac{a - c_1 - b(x_2 + x_3 + x_4)}{2b} - x_1 = 0$$

$$\frac{a - c_2 - b(x_1 + x_3 + x_4)}{2b} - x_2 = 0$$

$$\alpha\left(\frac{a - c_3 - b(x_1 + x_2 + x_4)}{2b}\right) - \alpha x_3 = 0$$

$$\beta x_4[a - 2bx_4 - b(x_1 + x_2 + x_3) - c_4] = 0$$

Solving the above equations, the values so obtained are:

$$x_1 = \frac{a + c_2 + c_3 - 3c_1}{4b}, x_2 = \frac{a + c_1 + c_3 - 3c_2}{4b}, x_3 = \frac{a + c_1 + c_2 - 3c_3}{4b}$$

Here

$$E_1 = \left(\frac{a + c_2 + c_3 - 3c_1}{4b}, \frac{a + c_1 + c_3 - 3c_2}{4b}, \frac{a + c_1 + c_2 - 3c_3}{4b}, 0\right) \tag{9}$$

is the boundary equilibrium point and

$$E_2 = \left(\frac{a + c_2 + c_3 + c_4 - 4c_1}{5b}, \frac{a + c_1 + c_3 + c_4 - 4c_2}{5b}, \frac{a + c_1 + c_2 + c_4 - 4c_3}{5b}, \frac{a + c_1 + c_2 + c_3 - 4c_4}{5b}\right) \tag{10}$$

is the Nash equilibrium point.
E_1 has non-negative coordinates if

$$\begin{cases} a + c_2 + c_3 > 3c_1 \\ a + c_1 + c_3 > 3c_2 \\ a + c_1 + c_2 > 3c_3 \end{cases}$$

E_2 has non-negative coordinates if

$$\begin{cases} a + c_2 + c_3 + c_4 > 4c_1 \\ a + c_1 + c_3 + c_4 > 4c_2 \\ a + c_1 + c_2 + c_4 > 4c_3 \\ a + c_1 + c_2 + c_3 > 4c_4 \end{cases}$$

With the purpose of checking the stability of the equilibrium points obtained above, the Jacobian matrix at the equilibrium points are calculated below. Also, the Eigen values of the Jacobian matrix are calculated in the succeeding steps. Nature of equilibrium points will determine the stability of the equilibrium points. Following is the Jacobian matrix:

$$J = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & 1-\alpha & -\frac{\alpha}{2} \\ -\beta b x_4 & -\beta b x_4 & -\beta b x_4 & 1+\beta(a-c_4-4bx_4-bx_1-bx_2-bx_3) \end{bmatrix}$$

At Boundary equilibrium point E_1 , Jacobian matrix is

$$J(E_1) = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & 1-\alpha & -\frac{\alpha}{2} \\ 0 & 0 & 0 & 1+\beta(a-4c_4+c_1+c_2+c_3) \end{bmatrix}$$

One of the Eigen values of $J(E_1)$ is $1+\frac{\beta}{4}(a+c_1+c_2+c_3-4c_4) > 0$ as $c_1+c_2+c_3 > 4c_4$,
So boundary equilibrium point is locally unstable.
At Nash equilibrium point E_2 , Jacobian matrix is

$$J(E_2) = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & 1-\alpha & -\frac{\alpha}{2} \\ -\beta b x_4^* & -\beta b x_4^* & -\beta b x_4^* & 1+\beta(a-c_4-4bx_4^*-bx_1^*-bx_2^*-bx_3^*) \end{bmatrix}$$

where x_1^*, x_2^*, x_3^* and x_4^* are the Nash equilibrium points.
Let γ be the Eigen values of , which are given by

$$\begin{vmatrix} -\gamma & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\gamma & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & 1-\alpha-\gamma & -\frac{\alpha}{2} \\ -\beta b x_4^* & -\beta b x_4^* & -\beta b x_4^* & 1+\beta(a-c_4-4bx_4^*-bx_1^*-bx_2^*-bx_3^*)-\gamma \end{vmatrix} = 0$$

So, Eigen values of above Jacobian matrix are roots of equation

$$\gamma^4 + A_1\gamma^3 + A_2\gamma^2 + A_3\gamma + A_4 = 0$$

, where

$$A_1 = 2-\alpha-\alpha\beta(a-c_4-4bx_4-bx_3-bx_1-bx_2)$$

$$A_2 = (1-\alpha)(1+\beta(a-c_4-4bx_4-bx_3-bx_1-bx_2)) - \frac{\alpha}{2} - \frac{1}{4} + \beta b x_4 \frac{\alpha}{4}$$

$$A_3 = -\frac{\beta b x_4}{2} - \beta b x_4 \alpha - \frac{\beta b x_4 \alpha}{8} + \left(\frac{\alpha}{2}+\frac{1}{4}\right)(1+\beta(a-c_4-4bx_4-bx_3-bx_1-bx_2)) + \frac{1}{4}$$

$$A_4 = -\frac{3\alpha}{8} - \frac{1}{4}(1+\beta(a-c_4-4bx_4-bx_3-bx_1-bx_2))$$

Schur Cohn Criterion shows that Conditions of local stability of Nash equilibrium points are:

$$\begin{cases} 1+A_1+A_2+A_3+A_4 > 0 \\ 1-A_1+A_2-A_3+A_4 > 0 \\ 1-A_4 > 0 \\ (1-A_4)(1-A_4^2) - A_2(1-A_4)^2 + (A_1-A_3)(A_3-A_1A_4) > 0 \end{cases}$$

5|Conclusion

This paper examined the profit maximizing strategies of a game with four players using linear demand and cost function.It revealed that boundary equilinrium point seems locally unstable given certain conditions.Nash equilibrium becomes stable for defined speed of adjustment and apadptive player experience steady effect on the market.It also established conditions of stability of Nash equilibrium points using Schur Cohn Criterion.

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Author Contribution

Bharti Kapoor: methodology, software, and editing. Shilpi Jain: conceptualization. Mehar Chand: writing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

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